On certain quasi-local spin-angular momentum expressions for large spheres near the null infinity

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The recently suggested quasi-local spin-angular momentum expressions, based on the Bramson superpotential and on the holomorphic or anti-holomorphic spinor fields, are calculated for large spheres near the future null infinity of asymptotically flat Einstein–Maxwell spacetimes. It is shown that although the expression based on the anti-holomorphic spinors is finite and unambiguously defined only in the center-of-mass frame (i.e. it diverges in general), the corresponding Pauli–Lubanski spin is always finite, free of gauge ambiguities, and is built only from the gravitational data. Thus it defines a gravitational spin expression at the future null infinity. The construction based on the holomorphic spinors diverges in presence of outgoing gravitational radiation. For stationary spacetimes both constructions reduce to the 'standard' expression.

1. Introduction

The energy-momentum and angular momentum *density* of matter fields is described by their energy-momentum tensor T^{ab}, whose contraction with a Killing field K_a gives a divergence free current $T^{ab}K_b$. The flux of this current on a compact, spacelike hypersurface Σ with boundary $\mathcal{S}:=\partial\Sigma$ is therefore conserved, i.e. depends only on the boundary $\mathcal{S}$ and independent of the rest of the hypersurface. In Minkowski spacetime the independent Killing 1-forms are the translations $K^{\mathbf{a}}_e:=\nabla_e x^{\mathbf{a}}$ and the rotations $K^{\mathbf{ab}}_e:=x^{\mathbf{a}}\nabla_e x^{\mathbf{b}}-x^{\mathbf{b}}\nabla_e x^{\mathbf{a}}$, where $x^{\mathbf{a}}$ are the standard Cartesian coordinates, $\mathbf{a},\mathbf{b}=0,...,3$. The corresponding conserved quantities are $P^{\mathbf{a}}_{\mathcal{S}}:=\int_\Sigma T^{ab}t_aK^{\mathbf{a}}_b{\rm d}\Sigma$ and $J^{\mathbf{ab}}_{\mathcal{S}}:=\int_\Sigma T^{ab}t_aK^{\mathbf{ab}}_b{\rm d}\Sigma$, where t_a is the future directed unit timelike normal to Σ and ${\rm d}\Sigma$ is the induced volume element on Σ. $P^{\mathbf{a}}_{\mathcal{S}}$ and $J^{\mathbf{ab}}_{\mathcal{S}}$ can be interpreted as the *quasi-local* energy-momentum and angular momentum of the matter fields associated with the 2-surface $\mathcal{S}$, respectively. If Σ extends to the spacelike infinity i^0 or to the future (or past) null infinity $\mathcal{I}^\pm$ of the Minkowski spacetime, then, provided the limits exists (by imposing certain fall-off and global integral conditions, e.g. parity conditions at the spacelike infinity), then they define the *global* energy-momentum and angular momentum.

In general relativity the usual fall-off conditions on the 3-metric and extrinsic curvature on a spacelike hypersurface ensure that the ADM energy and spatial momentum are finite and unambiguously defined [1,2]. However, to have finite *and* unambiguously defined angular momentum and center-of-mass additional conditions, e.g. explicit *parity conditions* on the three-metric h_{ab} and extrinsic curvature χ_{ab}, have to be imposed [2,3]. These are preserved by the evolution equations, ensure the functional differentiability of the Hamiltonian on the whole phase space, and yield unique Poincare structure both for the lapse-shift at infinity and the Hamiltonian. At spatial infinity the energy-momentum and relativistic angular momentum are the value of the functionally differentiable Hamiltonian, which are just the familiar ADM energy-momentum and spatial angular momentum, and the center-of-mass of Beig and Ó Murchadha [3]. Thus, although there are interesting open issues, e.g. whether the angular momentum measured at the spacelike infinity enters the Penrose inequality or not [4], the spatial infinity case is well understood.

At null infinity there is a generally accepted definition of the energy-momentum, which is the Bondi–Sachs energy-momentum. However, on the definition of angular momentum there is no consensus at all, and there are various suggestions for that: The constructions based on the Komar–Winicour–Tamburino expression are intended to be associated with any BMS vector field, and, in the special case of the boost-rotation vector fields, they can be interpreted as angular momentum [5-7]. The suggestion of Ashtekar and Streubel is based on symplectic techniques [8], which turned out later to be connected with the Winicour–Tamburino linkages [9]. The general form of other expressions are based on charge integrals of the curvature [10-16]: In particular, although the suggestion of Penrose [12] is based on the solutions of the 2-surface twistor equations and the concept of 'origin' is largely decoupled from the cuts of $\mathcal{I}^+$, it fits nicely to the symplectic structure of $\mathcal{I}^+$ [17-20]. Other remarkable suggestion in this class is due to Moreschi [13-16]. He considers (and, together with Dain, prove [15] the existence of) a foliation of $\mathcal{I}^+$ by special cuts (the 'nice cuts'), intended to model the 'system of rest frames'. Thus he is able to define not only a (supertranslation–ambiguity–free) angular momentum, but higher order moments of the 'gravitational field', too. In their classic paper Bergmann and Thomson [21] raises the idea that while the energy-momentum of gravity is connected with the spacetime diffeomorphisms, the angular momentum should be connected with its internal $O(1,3)$-symmetry. Thus the angular momentum should be analogous with the spin. This idea was formulated mathematically by Bramson [22-24], whose specific angular momentum expression was based on the superpotential derived from Hilbert's Lagrangian and the solutions of the asymptotic twistor equations. Recently, Katz and Lerer [25] could recover the Bondi–Sachs energy-momentum from the standard Noether analysis using asymptotically flat backgrounds, and suggested an expression for the angular momentum and center-of-mass. Interestingly enough, although these suggestions are based on different ideas and yield mathematically inequivalent definitions, all these resulting expressions bear some resemblance to each other. It is not clear whether the recent suggestion by Rizzi [26], based on a special foliation of the spacetime near the null infinity and reducing to the ADM angular momentum at the spatial infinity [27], also takes such a form.

At the quasi-local level the situation is even worse. Although the general framework of how the various quasi-local quantities should look like is more or less well understood (see e.g. [28]), only a few explicit definitions have been suggested for the energy-momentum *and* angular momentum. The first such suggestion, due to Penrose [12], is, however, not complete: In addition to the (symmetric) kinematical twistor $\mathtt{A}_{\alpha\beta}$ two additional twistors, a Hermitian metric twistor $\mathtt{h}_{\alpha\beta'}$ and the analog of the (skew, simple) infinity twistor $\mathtt{I}_{\alpha\beta}$, would be needed to reduce the ten complex components of $\mathtt{A}_{\alpha\beta}$ (the quasi-local quantities) to ten real ones, and to isolate four of them as the energy-momentum and six of them as the angular momentum. Although at the future null infinity these extra twistors exist and the construction works properly, reproducing the Bondi–Sachs energy-momentum and gives a definition for the angular momentum, it is not clear whether the additional twistors exist and the construction is viable quasi-locally or not. In principle the Brown–York approach also can yield both energy-momentum and angular momentum [29,30]. However, instead of a 4-covariant energy-momentum this approach yields separated energy and linear momentum, and it is not a priori clear whether they form a single Lorentz-covariant object. Furthermore, in lack of a unique prescription how the reference configuration should be find (imbedding of the 2-surface into a 3-plane [29], or into the light cone in Minkowski spacetime [31,32], or restrict the imbedding in some other way [33]), this construction is not complete either. To define Lorentz-covariant energy-momentum and angular momentum Ludvigsen and Vickers [34] used the Nester–Witten and the Bramson superpotentials, respectively, but their choice for the two spinor fields in the superpotential depends on the asymptotic structure of the spacetime. Thus their definition is not genuinely quasi-local. Genuinely quasi-local, manifestly Lorentz-covariant energy momenta, based on the Nester–Witten 2-form and either the holomorphic or the anti-holomorphic spinor fields, were suggested by Dougan and Mason [35], but they did not give any specific definition for the

angular momentum. Recently we suggested to complete the Dougan–Mason energy-momenta by a spin-angular momentum expression based on Bramson's superpotential, but, instead of the Ludvigsen–Vickers prescription for the spinor fields, we suggested to use holomorphic or anti-holomorphic spinor fields [36]. These have already been studied in various situations (*pp*-waves [36] and small spheres [37]). In the present paper these spin-angular momentum expressions will be calculated for large spheres near the future null infinity.

In [22] Bramson introduced his superpotential as the superpotential for the conserved $O(1,3)$-current $C^a_{\mathbf{ab}}$, derived from Hilbert's second order Lagrangian (considering that as a function of the tetrad field and the $O(1,3)$-connection 1-forms). However, he defined $C^a_{\mathbf{ab}}$ as the *partial* derivative of Hilbert's action with respect to the connection 1-forms, while in gauge theories the conserved currents corresponding to the internal gauge invariance are the *variational* derivatives of the particle action with respect to the connection 1-forms. In fact, this variational derivative is zero. Thus, in Section 2, first we show that Bramson's superpotential can be derived from Møller's tetrad action in a correct way, and then we review the status of our specific quasi-local spin-angular momentum expressions and provide the general formulae for the large sphere calculations. In Section 3. we review those structures of the future null infinity that we need, especially the BMS translations and rotations and their spinor form, and quote the asymptotic solution of the Einstein–Maxwell equations from [38]. Then, in Section 4, we calculate the anti-holomorphic spin-angular momentum for large spheres expanding that as a series of r^{-1}. We will see that although this is diverging linearly and its finite order part is ambiguous in general, the Pauli–Lubanski spin vector built from the anti-holomorphic Dougan–Mason energy-momentum and spin-angular momentum is always finite and well defined. To demonstrate this, we need the $O(r^{-1})$ accurate expansion of the Dougan–Mason energy-momentum. Since, however, this calculation has been done only for stationary spacetimes [39] (and apparently an r^{-1} order term was overlooked), we have to clarify the asymptotic behaviour of the energy-momentum, too. In Section 5 we show that in stationary spacetimes the anti-holomorphic construction reduces to the 'standard' expression. The holomorphic construction, calculated in Section 6, is diverging quadratically in presence of outgoing gravitational radiation, but gives the same 'standard' expression for stationary spacetimes. Finally, in the Appendix we discuss how the various spinor equations determine special spin frames on two-spheres, in particular, on round spheres and on large spheres near the future null infinity. We show how the BMS translations and rotations can be recovered from them. We found that, in addition to the usual representations, they can also be recovered from the solutions of the limit of the Dirac–Witten equations on the unit sphere cuts of $\mathcal{I}^+$, too. We treat the 2-surface twistor equations also in the traditional way instead of the almost exclusively followed conformal method.

Our conventions and notations are mostly those of [40]. In particular, we use the abstract index notations, and only the underlined and boldface indices take numerical values. The signature is -2, the Riemann- and Ricci tensors and the curvature scalar are defined by $-R^a{}_{bcd}X^bY^cZ^d := \nabla_Y(\nabla_Z X^a) - \nabla_Z(\nabla_Y X^a) - \nabla_{[Y,Z]}X^a$, $R_{ab} := R^e{}_{aeb}$ and $R := R_{ab}g^{ab}$, respectively. Thus Einstein's equations take the form $R_{ab} - \frac{1}{2}Rg_{ab} = -8\pi G T_{ab}$.

2. Bramson's superpotential and quasi-local spin-angular momenta for spheres

Let $\{E^a_{\mathbf{a}}, \vartheta^{\mathbf{a}}_a\}$, $\mathbf{a} = 0, ..., 3$, be an orthonormal dual frame field on an open domain $U \subset M$, and $\gamma^{\mathbf{a}}_{e\mathbf{b}} := \vartheta^{\mathbf{a}}_c \nabla_e E^c_{\mathbf{b}}$, the $O(1,3)$-connection 1-form on U. Then Møller's Lagrangian [41] takes the form $\mathcal{L} := \frac{1}{16\pi G}\sqrt{|g|}(R - 2\nabla_a(E^a_{\mathbf{a}}\eta^{\mathbf{ab}}E^c_{\mathbf{c}}\gamma^{\mathbf{c}}_{c\mathbf{b}})) = \frac{1}{16\pi G}\sqrt{|g|}(E^a_{\mathbf{a}}E^b_{\mathbf{b}} - E^a_{\mathbf{b}}E^b_{\mathbf{a}})\gamma^{\mathbf{a}}_{a\mathbf{c}}\gamma^{\mathbf{cb}}_b$, where $\eta_{\mathbf{ab}} := \mathrm{diag}(1,-1,-1,-1)$. Considering the Møller action as a functional both of the tetrad field and the connection 1-forms independently, $I_U[E^a_{\mathbf{a}}, \gamma^{\mathbf{a}}_{a\mathbf{b}}] := \int_U \mathcal{L}\mathrm{d}^4x$, for the $O(1,3)$ current we obtain

$$C^a_{\mathbf{ab}} := \frac{4\alpha}{\sqrt{|g|}}\frac{\delta I_U}{\delta\gamma^{\mathbf{ab}}_a} = -\frac{\alpha}{4\pi G}\nabla_b\big(E^a_{\mathbf{a}}E^b_{\mathbf{b}} - E^a_{\mathbf{b}}E^b_{\mathbf{a}}\big), \tag{2.1}$$

where α is some normalization constant, to be determined in some special situation. $C^a_{\mathbf{ab}}$ is identically conserved, and we call the corresponding superpotential 2-form, $W^{\mathbf{ac}}_{ac} := \frac{1}{4\pi G}\vartheta^{\mathbf{a}}_{[a}\vartheta^{\mathbf{c}}_{c]}$, the Bramson superpotential. The flux integral of this current for some spacelike submanifold Σ can be represented by the 2-surface integrals of the Bramson superpotential on its boundary $\mathcal{S} := \partial\Sigma$: $J^{\mathbf{ab}} := -\alpha\oint_{\mathcal{S}} W^{\mathbf{ab}}_{ab}\frac{1}{2}\varepsilon^{ab}{}_{cd} = -\alpha\frac{1}{2}\varepsilon^{\mathbf{ab}}{}_{\mathbf{cd}}\oint_{\mathcal{S}} W^{\mathbf{cd}}_{cd}$. If $\{E^a_{\mathbf{a}}\}$ is a constant orthonormal (i.e. a Cartesian) frame field in Minkowski spacetime, then $J^{\mathbf{ab}}$ is zero. To see this, it is enough to recall that in this case $\vartheta^{\mathbf{a}}_a = \nabla_a x^{\mathbf{a}}$, the gradient of the Cartesian coordinate functions, and hence $W^{\mathbf{ab}}_{cd}$ is an exact 2-form, whose integral on any closed orientable 2-surface is vanishing. Therefore, $J^{\mathbf{ab}}$ is a measure of how much the frame field $\{E^a_{\mathbf{a}}\}$ is distorted relative to a Cartesian basis in flat spacetime, i.e. a measure of the non-integrability of $\{E^a_{\mathbf{a}}\}$. (The current $\sqrt{|g|}C^{\mathbf{a}}_a := \delta I_U/\delta E^a_{\mathbf{a}}$ gives the *tensorial* energy-momentum expression $t^b{}_a$ for the gravitational 'field' found in [42,43] (and see also [44]): $C^{\mathbf{a}}_a = \vartheta^{\mathbf{a}}_b t^b{}_a$.)

If $\mathcal{S}$ is any closed orientable spacelike 2-surface in M and $\lambda^{\mathbf{A}}_A$, $\mathbf{A} = 0, 1$, is a pair of smooth spinor fields on $\mathcal{S}$ such that they form a normalized spin-frame, i.e. $\varepsilon^{AB}\lambda^{\mathbf{A}}_A\lambda^{\mathbf{B}}_B = \varepsilon^{\mathbf{AB}}$, where $\varepsilon^{\mathbf{AB}}$ is the antisymmetric Levi-Civita symbol, then $\vartheta^{\mathbf{a}}_a := \sigma^{\mathbf{a}}_{\mathbf{AB'}}\lambda^{\mathbf{A}}_A\bar\lambda^{\mathbf{B'}}_{A'}$ is an orthonormal 1-form field on $\mathcal{S}$, where $\sigma^{\mathbf{a}}_{\mathbf{AB'}}$ are the standard $SL(2,\mathbf{C})$ Pauli matrices. Since the Bramson superpotential depends on $\vartheta^{\mathbf{a}}_a$ algebraically, the integral $J^{\mathbf{ab}}$ can be expressed in terms of the normalized spin frame fields defined only on $\mathcal{S}$. $J^{\mathbf{ab}}$ is independent of the extension of $\{\lambda^{\mathbf{A}}_A\}$ off the 2-surface $\mathcal{S}$. Translating the tensor name indices of $J^{\mathbf{ab}}$ into spinor name indices, and defining its anti-self-dual part by $J^{\mathbf{AA'BB'}} =: \varepsilon^{\mathbf{AB}}\bar J^{\mathbf{A'B'}} + \varepsilon^{\mathbf{A'B'}}J^{\mathbf{AB}}$, we find that

$$J^{\mathbf{AB}} = \frac{\mathrm{i}\alpha}{8\pi G}\oint_{\mathcal{S}}\lambda^{\mathbf{A}}_{(A}\lambda^{\mathbf{B}}_{B)}\varepsilon_{A'B'}. \tag{2.2}$$

This is precisely the 2-surface integral of the spinorial Bramson 2-form [22]. Therefore, it has a natural Lagrangian interpretation in the sense that it is the spinor form of the superpotential of the conserved current derived from Møller's tetrad Lagrangian. In the present paper by quasi-local energy-momentum we mean an expression which is based on the 2-surface integral of the Nester–Witten 2-form:

$$P^{\mathbf{AB'}} := \frac{\mathrm{i}}{8\pi G}\oint_{\mathcal{S}}\big(\bar\lambda^{\mathbf{B'}}_{A'}\nabla_{BB'}\lambda^{\mathbf{A}}_A - \bar\lambda^{\mathbf{B'}}_{B'}\nabla_{AA'}\lambda^{\mathbf{A}}_B\big). \tag{2.3}$$

For any pair of spinor fields $\lambda^{\mathbf{A}}_A$ this defines a Lorentzian 4-vector, and it is natural to choose the spinor fields in (2.2) and (2.3) to be the same. (A more detailed discussion of these issues, and, in particular, the connection of these concepts and the current $C^{\mathbf{a}}_a$ will be given elsewhere [45].) The quasi-local Pauli–Lubanski spin vector will be defined in the standard way by

$$S_{\mathbf{AA'}} := \frac{1}{2}\varepsilon_{\mathbf{AA'BB'CC'DD'}}P^{\mathbf{BB'}}\big(\varepsilon^{\mathbf{C'D'}}J^{\mathbf{CD}} + \varepsilon^{\mathbf{CD}}\bar J^{\mathbf{C'D'}}\big) = \mathrm{i}\big(P_{\mathbf{AB'}}\bar J^{\mathbf{B'}}{}_{\mathbf{A'}} - P_{\mathbf{A'B}}J^{\mathbf{B}}{}_{\mathbf{A}}\big). \tag{2.4}$$

(If the quasi-local mass m, defined by $m^2 := \varepsilon_{\mathbf{AB}}\varepsilon_{\mathbf{A'B'}}P^{\mathbf{AA'}}P^{\mathbf{BB'}}$, is not zero, then the dimensionally correct definition of the Pauli–Lubanski spin is $\frac{1}{m}S_{\mathbf{AA'}}$.) To complete the construction of these quantities, however, the spin frame field $\lambda^{\mathbf{A}}_A$ must be specified on, and only on $\mathcal{S}$.

In the present paper we will assume that the spin frame is holomorphic, $\bar m^e\nabla_e\lambda_A = 0$, or anti-holomorphic, $m^e\nabla_e\lambda_A = 0$. Here m^a and $\bar m^a$ are the standard complex null vectors tangent to $\mathcal{S}$ and normalized by $\bar m^a m_a = -1$, by means of which the metric area-element on $\mathcal{S}$ is $-\mathrm{i}m_{[a}\bar m_{b]}$. In fact, one can show that in the generic case there are two holomorphic/anti-holomorphic spinor fields and they can be normalized, provided $\mathcal{S}$ is homeomorphic to S^2. Thus $P^{\mathbf{AB'}}$ and $J^{\mathbf{AB}}$ that we will study here are the two energy-momenta of Dougan and Mason [35], and the two spin-angular momenta that we introduced in [36],

respectively. Then the formers can also be written as $P^{\mathbf{AB}'}=-\gamma\frac{\mathrm{i}}{4\pi G}\oint_{\mathcal{S}}(\rho'\lambda^{\mathbf{A}}_0\bar\lambda^{\mathbf{B}'}_{0'}+\rho\lambda^{\mathbf{A}}_1\bar\lambda^{\mathbf{B}'}_{1'})m_{[a}\bar m_{b]}$, where $\gamma=-1$ for the holomorphic, and $\gamma=+1$ for the anti-holomorphic spinor fields. In Minkowski spacetime, for example, the independent holomorphic spinors are anti-holomorphic too, and they are just the constant spinor fields restricted to $\mathcal{S}$, when obviously $P^{\mathbf{AB}'}=0$ and, as we saw above, $J^{\mathbf{AB}}$ is also vanishing [36,37]. If $\varepsilon^A_{\underline{A}}:=\{o^A,\iota^A\}$, $\underline{A}=0,1$, is a standard GHP spin frame adapted to the 2-surface, $\varepsilon^{\underline{A}}_A:=\{-\iota_A,o_A\}$ is its dual basis and the spinor components with respect to this frame are defined by $\lambda^{\mathbf{A}}_{\underline{A}}=:\varepsilon^A_{\underline{A}}\lambda^{\mathbf{A}}_A$, then the condition of holomorphy can be written as $\eth'\lambda^{\mathbf{A}}_1+\sigma'\lambda^{\mathbf{A}}_0=0$ and $\eth'\lambda^{\mathbf{A}}_0+\rho\lambda^{\mathbf{A}}_1=0$; and the condition of anti-holomorphy is equivalent to $\eth\lambda^{\mathbf{A}}_0+\sigma\lambda^{\mathbf{A}}_1=0$ and $\eth\lambda^{\mathbf{A}}_1+\rho'\lambda^{\mathbf{A}}_0=0$. Thus boldface capital indices are referring to a basis in the space of solutions, while the underlined capital indices to the GHP spin frame. For example, for a round sphere [39] of radius r the two linearly independent anti-holomorphic and holomorphic spinor fields are given by (A.2.2) and (A.3.2) of the Appendix, respectively. Substituting them into (2.2) we get zero, as could be expected in a spherically symmetric system.

It is known that the anti-holomorphic Dougan–Mason energy-momentum is a future directed non-spacelike vector if $\mathcal{S}$ is the boundary of some compact spacelike hypersurface Σ, the matter fields satisfy the dominant energy condition on Σ, and $\mathcal{S}$ is convex in the weak sense that $\rho'\geq 0$ [35]. (For the holomorphic construction the analogous convexity condition is $\rho\leq 0$.) Furthermore, if the dominant energy condition holds on the whole domain of dependence $D(\Sigma)$ of Σ then $P^{\mathbf{AB}'}$ is vanishing if and only if $D(\Sigma)$ is flat, and this is also equivalent to the existence of two constant spinor fields on $\mathcal{S}$ with respect to the covariant derivative Δ_e (see the Appendix); and $P^{\mathbf{AB}'}$ is null if and only if $D(\Sigma)$ has a *pp*-wave metric and the matter is pure radiation, which is also equivalent to the existence of one constant spinor field on $\mathcal{S}$ [46,47]. $J^{\mathbf{AB}}$ has already been calculated for (axi-symmetric) $\mathcal{S}$ bounding a *pp*-wave Cauchy development and it was shown that the Pauli–Lubanski spin $S_{\mathbf{AB}'}$ is proportional to the (null) $P_{\mathbf{AB}'}$ [36].

We have already calculated (2.2) for the Ludvigsen–Vickers-, the holomorphic- and the anti-holomorphic spinors for small spheres $\mathcal{S}_r$ of radius r with respect to an observer t^a at a point $o\in M$ [37]. Considering this to be a function of the radius, $\underset{r}{J}^{\mathbf{AB}}$, it can be expanded as a power series of r. The leading term is $-\frac{4\alpha}{3}\pi r^4T_{AA'BB'}t^{AA'}t^{B'E}\varepsilon^{BF}\mathcal{E}^{(\mathbf{A}}_E\mathcal{E}^{\mathbf{B})}_F$ for the Ludvigsen–Vickers and the holomorphic spinors, whilst that is its negative for the anti-holomorphic spinors. In vacuum the leading term is $-\frac{4\alpha}{45G}r^6T_{AA'BB'CC'DD'}t^{AA'}t^{BB'}t^{CC'}t^{D'E}\varepsilon^{DF}\mathcal{E}^{(\mathbf{A}}_E\mathcal{E}^{\mathbf{B})}_F$ in all cases, where T_{abcd} is the Bel–Robinson tensor and $\{\mathcal{E}^A_{\mathbf{A}},\mathcal{E}^{\mathbf{A}}_A\}$ is the dual Cartesian spin frame at o; i.e. $E^a_{\mathbf{a}}=\sigma^{\mathbf{AA}'}_{\mathbf{a}}\mathcal{E}^A_{\mathbf{A}}\bar{\mathcal{E}}^{A'}_{\mathbf{A}'}$ and $\vartheta^{\mathbf{a}}_a=\sigma^{\mathbf{a}}_{\mathbf{AA}'}\mathcal{E}^{\mathbf{A}}_A\bar{\mathcal{E}}^{\mathbf{A}'}_{A'}$ form an orthonormal dual frame at o. Thus the 'pure gravitational field' itself does not seem to contribute to the spin-angular momentum in r^4 order. On the other hand, for the leading term in the similar expansion of the quasi-local (anti-self-dual) angular momentum of the matter fields in Minkowski spacetime we get $\frac{4}{3}\pi r^4T_{AA'BB'}t^{AA'}t^{B'E}\varepsilon^{BF}\mathcal{E}^{(\mathbf{A}}_E\mathcal{E}^{\mathbf{B})}_F$. The coefficient $rt^{A'E}\varepsilon^{AF}\mathcal{E}^{\mathbf{A}}_{(E}\mathcal{E}^{\mathbf{B}}_{F)}$ is just an average of the approximate (and in Minkowski spacetime the exact) anti-self-dual boost-rotation Killing vector that vanishes at o, where the average is taken on the unit sphere $\mathcal{S}$. Therefore, identifying this a.s.d. angular momentum of the matter fields with the r^4 order *holomorphic* $\underset{r}{J}^{\mathbf{AB}}$, we obtain $\alpha=-1$, but identifying with the *anti-holomorphic* $\underset{r}{J}^{\mathbf{AB}}$ we get $\alpha=+1$. Thus there are two equally reasonable conventions: Requiring the r^4 order terms to coincide with that of the angular momentum of the matter fields, whenever the sign α is present in the basic definition (2.1) and in the two cases the r^6 order leading terms of $\underset{r}{J}^{\mathbf{AB}}$ in vacuum are just minus the others; or to fix α once and for all for both constructions in the same way (e.g. by $\alpha=-1$ as we did in [36,37]). We will see that in the latter case $\alpha=1$ would be a slightly more natural choice because the *anti-holomorphic* rather than the holomorphic constructions, both for the energy-momentum and spin-angular momentum, fit to the structure of the *future* null infinity. However, keeping in mind these observations, we retain our previous conventions, simply to facilitate the comparison with the previous works.

In the rest of the paper the 2-surface will be assumed to be a large sphere of radius r near the future null infinity in the sense defined in section 3 below, and the corresponding spin-angular momentum will be denoted

by $J_r^{\mathbf{AB}}$. Then, according to the general philosophy of the large sphere calculations [38,39], we expand the spinor components $\lambda^{\mathbf{A}}_{\underline{A}}$, the GHP spin coefficients and the area element $\mathrm{d}\mathcal{S}_r$ as series of $\frac{1}{r}$. The expansion coefficients as functions of the remaining coordinates depend on the actual construction (holomorphic or anti-holomorphic). If therefore $\lambda^{\mathbf{A}}_{\underline{A}} =: \lambda^{\mathbf{A}}_{\underline{A}}{}^{(0)} + \frac{1}{r}\lambda^{\mathbf{A}}_{\underline{A}}{}^{(1)} + \frac{1}{r^2}\lambda^{\mathbf{A}}_{\underline{A}}{}^{(2)} + ...$ and $\mathrm{d}\mathcal{S}_r =: r^2(1 + \frac{1}{r}s^{(1)} + \frac{1}{r^2}s^{(2)} + ...)\mathrm{d}\mathcal{S}$, where $\mathrm{d}\mathcal{S}$ is the area element on the unit sphere $\mathcal{S}$, then (2.2) takes the form*

$$
\begin{aligned}
J_r^{\mathbf{AB}} = \frac{1}{8\pi G}\oint_{\mathcal{S}} \Big\{ & r^2\Big(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)} + \lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\Big) + r\Big(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(1)} + \lambda_0^{\mathbf{A}(1)}\lambda_1^{\mathbf{B}(0)} + \lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(1)} + \lambda_1^{\mathbf{A}(1)}\lambda_0^{\mathbf{B}(0)}\Big) + \\
& + \Big(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(2)} + \lambda_0^{\mathbf{A}(1)}\lambda_1^{\mathbf{B}(1)} + \lambda_0^{\mathbf{A}(2)}\lambda_1^{\mathbf{B}(0)} + \lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(2)} + \lambda_1^{\mathbf{A}(1)}\lambda_0^{\mathbf{B}(1)} + \lambda_1^{\mathbf{A}(2)}\lambda_0^{\mathbf{B}(0)}\Big) + ...\Big\}\times \\
& \times\Big(1 + \frac{1}{r}s^{(1)} + \frac{1}{r^2}s^{(2)} + ...\Big)\mathrm{d}\mathcal{S}.
\end{aligned}
\tag{2.5}
$$

Therefore the $r\to\infty$ limit of $J_r^{\mathbf{AB}}$ exists iff the r^2 and r order terms integrate to zero. Similarly,

$$
\begin{aligned}
P_r^{\mathbf{AB}'} = \frac{\gamma}{4\pi G}\oint_{\mathcal{S}} \Big\{ & r\Big(\rho'^{(1)}\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)} + \rho^{(1)}\lambda_1^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(0)}\Big) + \\
& + \Big(\rho'^{(1)}\big(\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(1)} + \lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\big) + \rho'^{(2)}\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)} + \rho^{(1)}\big(\lambda_1^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(1)} + \\
& \quad + \lambda_1^{\mathbf{A}(1)}\bar\lambda_{1'}^{\mathbf{B}'(0)}\big) + \rho^{(2)}\lambda_1^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(0)}\Big) + \\
& + \frac{1}{r}\Big(\rho'^{(1)}\big(\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(2)} + \lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(1)} + \lambda_0^{\mathbf{A}(2)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\big) + \rho'^{(2)}\big(\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(1)} + \lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\big) + \\
& \quad + \rho'^{(3)}\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)} + \rho^{(1)}\big(\lambda_1^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(2)} + \lambda_1^{\mathbf{A}(1)}\bar\lambda_{1'}^{\mathbf{B}'(1)} + \lambda_1^{\mathbf{A}(2)}\bar\lambda_{1'}^{\mathbf{B}'(0)}\big) + \\
& \quad + \rho^{(2)}\big(\lambda_1^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(1)} + \lambda_1^{\mathbf{A}(1)}\bar\lambda_{1'}^{\mathbf{B}'(0)}\big) + \rho^{(3)}\lambda_1^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(0)}\Big) + ...\Big\}\times \\
& \times\Big(1 + \frac{1}{r}s^{(1)} + \frac{1}{r^2}s^{(2)} + ...\Big)\mathrm{d}\mathcal{S},
\end{aligned}
\tag{2.6}
$$

which has a finite limit iff its r order term integrates to zero.

3. The geometry of large spheres

We assume that the spacetime is future weakly asymptotically simple [40], and let $(u, r, \zeta, \bar\zeta)$ be the standard Bondi coordinate system in a neighbourhood of the future null infinity $\mathscr{I}^+$ based on a cut $\mathcal{S}_0$ of $\mathscr{I}^+$ (the 'origin') (see also [48,49]). However, following [38], we adapt the null tetrad (and the spin frame) to the foliation of the outgoing null hypersurfaces by the 2-spheres $\mathcal{S}_{(u,r)} := \{u = \text{const}, \ r = \text{const}\}$, which are called the *large spheres*. (In particular, the origin $\mathcal{S}_0$ above can be thought of as the $r\to\infty$ limit of the large spheres $\mathcal{S}_{(0,r)}$.) Explicitly, let $o_A\bar o_{A'} := l_a := \nabla_a u$, which is geodesic, and impose the condition $l^a\nabla_a o_B = 0$. Let $n^a = \iota^A\bar\iota^{A'}$ be the other future directed null normal to $\mathcal{S}_{(u,r)}$, normalized by $n^a l_a = 1$. Then $m^a = o^A\bar\iota^{A'}$ and $\bar m^a = \iota^A\bar o^{A'}$ are tangent to $\mathcal{S}_{(u,r)}$, and the conditions above fix the basis. With this choice for the tetrad we have the following restrictions on the GHP spin coefficients $\kappa = \varepsilon = \rho - \bar\rho = \tau + \bar\beta' - \beta = \rho' - \bar\rho' = \tau' - \beta' + \bar\beta = 0$. Therefore, $\text{þ} = (\partial/\partial r)$, $\eth f = (\delta - (p-q)\beta - q\tau)f$ and $\eth' f = (\bar\delta + (p-q)\bar\beta - p\bar\tau)f$ for any scalar f of type (p,q),

* We use two different notations for the expansion coefficients: $f^{(k)}$ (i.e. when the index k is between parentheses) denotes the coefficient of r^{-k} in the expansion, which may turn out later to be zero. On the other hand, as is usual in the relevant literature, $f^{\,k}$ will denote the $(k+1)$th *nonvanishing* expansion coefficient of the function $f = f(\frac{1}{r})$. However, for a function f both $f^{\,k}$ and $f^{(k)}$ will never appear.

where $\delta := m^a\nabla_a =: P(\partial/\partial\bar\zeta)+Q(\partial/\partial\zeta)$, and ζ, $\bar\zeta$ are the standard complex stereographic coordinates on the 2-sphere. In particular, on the unit sphere in the Minkowski spacetime the edth and edth-prime operators take the form ${}_0\eth f := \frac{1}{\sqrt{2}}(1+\zeta\bar\zeta)(\partial f/\partial\bar\zeta) + \frac{1}{2\sqrt{2}}(p-q)\zeta f$ and ${}_0\eth' f := \frac{1}{\sqrt{2}}(1+\zeta\bar\zeta)(\partial f/\partial\zeta) - \frac{1}{2\sqrt{2}}(p-q)\bar\zeta f$, which will be used in the subsequent calculations. Overdot will denote partial derivative with respect to u.

In the coordinate system $(u,w,\zeta,\bar\zeta)$, where $w := r^{-1}$ and the future null infinity is given by $w=0$, the BMS vector fields have the form $k^a = (H + (c^{\mathbf{i}}+\bar c^{\mathbf{i}})t_{\mathbf{i}}\,u)(\frac{\partial}{\partial u})^a + c^{\mathbf{i}}\xi_{\mathbf{i}}(\frac{\partial}{\partial\zeta})^a + \bar c^{\mathbf{i}}\bar\xi_{\mathbf{i}}(\frac{\partial}{\partial\bar\zeta})^a + O(r^{-1})$, where H is an arbitrary real valued function of ζ and $\bar\zeta$, $c^{\mathbf{i}}\in\mathbf{C}$, $\mathbf{i}=1,2,3$, and $t_1 := -(\zeta+\bar\zeta)(1+\zeta\bar\zeta)^{-1}$, $t_2 := \mathrm{i}(\zeta-\bar\zeta)(1+\zeta\bar\zeta)^{-1}$, $t_3 := (1-\zeta\bar\zeta)(1+\zeta\bar\zeta)^{-1}$ and $\xi_1 := (1-\zeta^2)$, $\xi_2 := \mathrm{i}(1+\zeta^2)$ and $\xi_3 := 2\zeta$. For the generators of the BMS supertranslations $c^{\mathbf{i}}=0$, and, in particular, the independent BMS translations take the form $t^a_{\mathbf{a}} = t_{\mathbf{a}}(\frac{\partial}{\partial u})^a + O(r^{-1})$, $\mathbf{a}=0,...,3$, where $t_0 := 1$ and $t_{\mathbf{i}}$ for $\mathbf{i}=1,2,3$ are given explicitly above. The functions $t_{\mathbf{a}}$ can be written as $t_{\mathbf{a}} = \sigma^{\mathbf{AB}'}_{\mathbf{a}}\tau_{\mathbf{A}}\bar\tau_{\mathbf{B}'}$, where $\tau_0 := \exp(\mathrm{i}\alpha)\sqrt[4]{2}(1+\zeta\bar\zeta)^{-\frac{1}{2}}$ and $\tau_1 := -\exp(\mathrm{i}\alpha)\sqrt[4]{2}\zeta(1+\zeta\bar\zeta)^{-\frac{1}{2}}$, and $\exp(\mathrm{i}\alpha)$ is an unspecified phase. In Minkowski spacetime the standard constant orthonormal frame field $\{E^a_{\mathbf{a}}\}$ (i.e. the translational Killing vectors $K^a_{\mathbf{a}}$) has precisely this asymptotic form, and if the phase $\exp(\mathrm{i}\alpha)$ is chosen to be $-\mathrm{i}$ then the functions $\tau_{\mathbf{A}}$ are just $\mathcal{E}^A_{\mathbf{A}}o_A$, the contractions of the Cartesian spin frame with the GHP spin vector o_A. In fact, in terms of the GHP spin frame $\varepsilon^A_{\underline{A}} = \{o^A,\iota^A\}$ the Cartesian spin frame $\mathcal{E}^A_{\mathbf{A}} = \{O^A, I^A\}$ has the form

$$O^A = -\frac{\mathrm{i}}{\sqrt[4]{2}}\frac{1}{\sqrt{1+\zeta\bar\zeta}}\Big(\bar\zeta o^A + \sqrt{2}\iota^A\Big), \qquad I^A = -\frac{\mathrm{i}}{\sqrt[4]{2}}\frac{1}{\sqrt{1+\zeta\bar\zeta}}\Big(o^A - \sqrt{2}\zeta\iota^A\Big). \tag{3.1}$$

Thus the BMS translations above are properly normalized. In the conformal approach (see, e.g. [40]) the GHP spin frame is rescaled by $\hat o^A = \Omega^{-1}o^A$ and $\hat\iota^A = \iota^A$, thus expressing o^A and ι^A by the non-physical spin vectors $\hat o^A$ and $\hat\iota^A$ in (3.1), for the asymptotic behaviour of the Cartesian spin frame (i.e. the spinor constituents of the translation Killing vectors) we directly obtain $\mathcal{E}^A_{\mathbf{A}} = \tau_{\mathbf{A}}\hat\iota^A + O(\Omega)$, where the $\exp(\mathrm{i}\alpha) = -\mathrm{i}$ choice was made and the conformal factor is $\Omega = r^{-1}$. To find the proper normalization of the boost-rotation BMS generators too, let us consider the Minkowski spacetime again. The anti-self-dual part in the name indices of the boost-rotation Killing vectors $K^a_{\mathbf{ab}}$ of the Minkowski spacetime is $K^a_{\mathbf{AB}} = \frac{1}{\sqrt{2}}\sigma^{\mathbf{i}}_{\mathbf{AB}}(K^a_{0\mathbf{i}} + \frac{\mathrm{i}}{2}\varepsilon_{\mathbf{i}}{}^{\mathbf{jk}}K^a_{\mathbf{jk}})$, where $\mathbf{i},\mathbf{j},\mathbf{k}=1,2,3$ and $\sigma_{\mathbf{i}}{}^{\mathbf{A}}{}_{\mathbf{B}} := \sqrt{2}\sigma_{\mathbf{i}}{}^{\mathbf{A}}{}_{\mathbf{B}'}\sigma_0^{\mathbf{B}'}{}_{\mathbf{B}}$, the standard $SU(2)$-Pauli matrices, and $\varepsilon_{\mathbf{ijk}}$ is the alternating Levi-Civita symbol, whose indices are moved by the constant (negative definite) metric $\eta_{\mathbf{ij}} := \mathrm{diag}\,(-1,-1,-1)$. They yield on $\mathcal{I}^+$ the BMS vector fields $k^a_{\mathbf{AB}} := K^a_{\mathbf{AB}}|_{\mathcal{I}^+} = \frac{1}{\sqrt{2}}\sigma^{\mathbf{i}}_{\mathbf{AB}}(ut_{\mathbf{i}}(\frac{\partial}{\partial u})^a + \xi_{\mathbf{i}}(\frac{\partial}{\partial\zeta})^a) + O(r^{-1}) = \sigma^{\mathbf{i}}_{\mathbf{AB}}(\frac{1}{\sqrt{2}}ut_{\mathbf{i}}\hat\iota^A\bar{\hat\iota}^{A'} + \xi_{\mathbf{i}}(1+\zeta\bar\zeta)^{-1}\hat\iota^A\bar{\hat o}^{A'}) + O(\Omega)$. Note that the functions $\xi_{\mathbf{i}}(1+\zeta\bar\zeta)^{-1}$ can be written as $\xi_{\mathbf{i}}(1+\zeta\bar\zeta)^{-1} = -\sigma^{\mathbf{AB}}_{\mathbf{i}}\tau_{\mathbf{A}}\tau_{\mathbf{B}}$, where the phase in $\tau_{\mathbf{A}}$ has been (and in the rest of the paper will be) chosen as $\exp(\mathrm{i}\alpha) = -\mathrm{i}$. Therefore, apart from the (in general super) translation content, the generator of the anti-self-dual rotations at $\mathcal{I}^+$ is $\sigma^{\mathbf{i}}_{\mathbf{AB}}\xi_{\mathbf{i}}(1+\zeta\bar\zeta)^{-1} = -\tau_{(\mathbf{A}}\tau_{\mathbf{B})}$, i.e. minus the symmetrized product of the functions $\tau_{\mathbf{A}}$. In the Appendix we discuss how the generators of the translations and of the anti-self-dual rotations can be recovered from the solutions of various spinor equations on the cuts of $\mathcal{I}^+$.

Suppose that, at least in a neighbourhood of $\mathcal{I}^+$, the only matter field that we have is the electromagnetic field, represented by the components of the Maxwell spinor (see e.g. [48,49]). Imposing slightly stronger fall-off conditions than that coming from the definition of the future weak asymptotic simplicity, viz. assuming that $\psi_0 = r^{-5}\psi_0{}^0 + r^{-6}\psi_0{}^1 + O(r^{-7})$ and $\phi_0 = r^{-3}\phi_0{}^0 + r^{-4}\phi_0{}^1 + O(r^{-5})$, Shaw found the asymptotic solution of the Einstein–Maxwell equations [38]. We need this solution with accuracy $O(r^{-3})$. In particular,

$$P = \frac{1}{\sqrt{2}}\big(1+\zeta\bar\zeta\big)\Big(\frac{1}{r} + \frac{1}{r^3}\sigma^0\bar\sigma^0\Big) + O(r^{-5}), \tag{3.2}$$

$$Q = \frac{1}{\sqrt{2}}\big(1+\zeta\bar\zeta\big)\Big(-\frac{1}{r^2}\sigma^0 + \frac{1}{r^4}\big(\frac{1}{6}\psi_0{}^0 - (\sigma^0)^2\bar\sigma^0\big)\Big) + O(r^{-5}). \tag{3.3}$$

This implies that the area element of the large sphere of radius r is $\mathrm{d}\mathcal{S}_{(u,r)}=r^2(1-r^{-2}\sigma^0\bar\sigma^0+O(r^{-4}))\mathrm{d}\mathcal{S}$; i.e. in equation (2.4) $s^{(1)}=0$ and $s^{(2)}=-\sigma^0\bar\sigma^0$. The spin coefficients with definite (p,q) type are

$$\sigma=\frac{1}{r^2}\sigma^0+O(r^{-4}), \tag{3.4}$$

$$\rho=-\frac{1}{r}-\frac{1}{r^3}\sigma^0\bar\sigma^0+O(r^{-5}), \tag{3.5}$$

$$\begin{aligned}\sigma'=&-\frac{1}{r}\dot{\bar\sigma}^0-\frac{1}{r^2}\Bigl(\frac{1}{2}\bar\sigma^0-{}_0\eth'\,{}_0\eth\bar\sigma^0\Bigr)-\frac{1}{r^3}\Bigl(\sigma^0\bar\sigma^0\dot{\bar\sigma}^0+\frac{1}{2}\bar\sigma^0\psi_2{}^0+\frac{1}{2}\,{}_0\eth'\bar\psi_{1'}{}^0+\\&+\bar\sigma^0\bigl({}_0\eth^2\bar\sigma^0+{}_0\eth'^2\sigma^0\bigr)+{}_0\eth'\bar\sigma^0\,{}_0\eth'\sigma^0-{}_0\eth\bar\sigma^0\,{}_0\eth\bar\sigma^0-G\phi_2{}^0\bar\phi_{0'}{}^0\Bigr)+O(r^{-4}),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\rho'=&\frac{1}{2r}+\frac{1}{r^2}\Bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\Bigr)-\frac{1}{r^3}\Bigl({}_0\eth\bar\sigma^0\,{}_0\eth'\sigma^0-\frac{1}{2}\sigma^0\bar\sigma^0+\sigma^0\,{}_0\eth'\,{}_0\eth\bar\sigma^0+\\&+\bar\sigma^0\,{}_0\eth\,{}_0\eth'\sigma^0+\frac{1}{2}\,{}_0\eth'\psi_1{}^0+\frac{1}{2}\,{}_0\eth\bar\psi_{1'}{}^0-2G\phi_1{}^0\bar\phi_{1'}{}^0\Bigr)+O(r^{-4}),\end{aligned} \tag{3.7}$$

$$\tau=\frac{1}{r^2}\,{}_0\eth'\sigma^0-\frac{1}{r^3}\Bigl(2\sigma^0\,{}_0\eth\bar\sigma^0+\psi_1{}^0\Bigr)+O(r^{-4}); \tag{3.8}$$

while the spin coefficient representing the GHP connection 1-form is

$$\beta=-\frac{1}{r}\frac{1}{2\sqrt{2}}\zeta-\frac{1}{r^2}\frac{1}{2\sqrt{2}}\bar\zeta\sigma^0-\frac{1}{r^3}\Bigl(\frac{1}{2\sqrt{2}}\zeta\sigma^0\bar\sigma^0+\frac{1}{2}\psi_1{}^0+\sigma^0\,{}_0\eth\bar\sigma^0\Bigr)+O(r^{-4}). \tag{3.9}$$

Finally, for the Weyl and Maxwell spinor components one has the familiar peeling: $\psi_n=r^{n-5}\psi_n{}^0+O(r^{n-6})$, $n=0,...,4$, and $\phi_n=r^{n-3}\phi_n{}^0+O(r^{n-4})$, $n=0,1,2$, where $\psi_4{}^0=-\ddot{\bar\sigma}^0$, $\psi_3{}^0=-{}_0\eth\dot{\bar\sigma}^0$ and $\psi_2{}^0-\bar\psi_{2'}{}^0=\bar\sigma^0\dot\sigma^0-\sigma^0\dot{\bar\sigma}^0+{}_0\eth'^2\sigma^0-{}_0\eth^2\bar\sigma^0$.

The spacetime will be called stationary, if it admits a timelike Killing vector field, at least on an open neighbourhood of $\mathcal{I}^+$, which can be extended to $\mathcal{I}^+$ into a BMS translation [48,38]. Then there is a cut, chosen to be the new origin, whose asymptotic shear is vanishing, and the asymptotic shear on the cut given by the supertranslation $S:\mathcal{S}\to\mathbf{R}$ with respect to the new origin is $\sigma^0=-{}_0\eth^2S$. Then, after an appropriate translation of the new origin, the asymptotic value of the Weyl and Maxwell spinor components that we need in what follows on such a cut take the form $\psi_1{}^0=-3G\,{}_0\eth(MS+\mathrm{i}J)$, $\psi_2{}^0=-GM$ and $\phi_1{}^0=\frac{1}{2}(e+\mathrm{i}\mu)$. Here M, e and μ are real constants, interpreted as the total mass, the total electric charge and the total magnetic charge, respectively, and J is a real function with structure $J=\sum_{m=-1}^{m=1}J_mY_{1,m}$ for some constants J_0 and $J_{\pm1}$, where $Y_{1,m}$ are the standard $j=1$ spherical harmonics. Rewriting J by the familiar polar coordinates (θ,ϕ) (defined by $\zeta=:\cot\frac{\theta}{2}\exp(\mathrm{i}\phi)$) into the form $J=j_1\sin\theta\cos\phi+j_2\sin\theta\sin\phi+j_3\cos\theta$, the real constants $j_{\mathbf{i}}=(j_1,j_2,j_3)$ defined in this way are interpreted as the components of the total spatial angular momentum of the stationary solution. In particular, for the Kerr–Newman solution $j_1=j_2=0$ and $j_3=Ma$, and hence on the shear-free $u=$ const. cuts $\psi_1{}^0=-\frac{3}{\sqrt{2}}\mathrm{i}GMa\sin\theta\exp(\mathrm{i}\phi)$. (Its apparent deviation from the expression given in [17,38] by the factor $-\exp(\mathrm{i}\phi)$ is a consequence of the different choices for the holomorphic coordinates on $\mathcal{S}$: Our choice is $\bar\zeta$ [even if spherical polar coordinates are present], while in [17,38] it is $-\log\bar\zeta=\log\tan\frac{\theta}{2}+\mathrm{i}\phi$.)

4. The anti-holomorphic spin-angular momentum

Substituting the expansion $\lambda_{\underline{A}}=:\lambda_{\underline{A}}{}^{(0)}+\frac{1}{r}\lambda_{\underline{A}}{}^{(1)}+\frac{1}{r^2}\lambda_{\underline{A}}{}^{(2)}+...$ of the spinor components and the expressions (3.2-9) for the functions P and Q and the spin coefficients into the equations defining the anti-holomorphic spinor fields on $\mathcal{S}_{(u,r)}$, we obtain the following hierarchy of equations

$$
\begin{aligned}
&{}_0\eth\lambda_1{}^{(0)}+\frac{1}{2}\lambda_0{}^{(0)}=0, && (4.1.a)\\
&{}_0\eth\lambda_0{}^{(0)}=0, && (4.1.b)\\
&{}_0\eth\lambda_1{}^{(1)}+\frac{1}{2}\lambda_0{}^{(1)}=-\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\big)\lambda_0{}^{(0)}, && (4.2.a)\\
&{}_0\eth\lambda_0{}^{(1)}=0, && (4.2.b)\\
&{}_0\eth\lambda_1{}^{(2)}+\frac{1}{2}\lambda_0{}^{(2)}=\sigma^0\,{}_0\eth'\lambda_1{}^{(1)}-\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\big)\lambda_0{}^{(1)}+\big(\sigma^0\,{}_0\eth\bar\sigma^0+\frac{1}{2}\psi_1{}^0\big)\lambda_1{}^{(0)}+\\
&\quad+\big({}_0\eth\bar\sigma^0\,{}_0\eth'\sigma^0+\sigma^0\,{}_0\eth'{}_0\eth\bar\sigma^0+\bar\sigma^0\,{}_0\eth\,{}_0\eth'\sigma^0+\frac{1}{2}\,{}_0\eth'\psi_1{}^0+\frac{1}{2}\,{}_0\eth\bar\psi_{1'}{}^0-2G\phi_1{}^0\bar\phi_{1'}{}^0\big)\lambda_0{}^{(0)}, && (4.3.a)\\
&{}_0\eth\lambda_0{}^{(2)}=\sigma^0\big({}_0\eth'\lambda_0{}^{(1)}-\lambda_1{}^{(1)}\big)-\big(\sigma^0\,{}_0\eth\bar\sigma^0+\frac{1}{2}\psi_1{}^0\big)\lambda_0{}^{(0)}. && (4.3.b)
\end{aligned}
$$

Thus by (4.1.a-b) the zeroth order spinor components are just the components of the constant spinors of Minkowski spacetime, and hence, in addition to (4.1.a-b), they satisfy ${}_0\eth'\lambda_1{}^{(0)}=0$ and ${}_0\eth'\lambda_0{}^{(0)}=\lambda_1{}^{(0)}$, too. Therefore, there are precisely two solutions $\lambda^{\mathbf{A}\,(0)}_{\underline{A}}$, $\mathbf{A}=0,1$, of (4.1.a-b), and we choose them to be given explicitly by (A.2.2) with $\rho'=\frac{1}{2r}$ (see the Appendix).

Since the left hand side of (4.1), (4..2) and (4.3) are the same, the solution in each order will be the sum of a particular solution and the general zeroth order (i.e. the constant) solution. Therefore, the general r^{-2} accurate anti-holomorphic spinor fields form a six rather than the expected two complex dimensional space. In fact, in the small sphere calculations [37] we had similar spurious solutions, due to the fact that there is no natural isomorphism between the space of the anti-holomorphic spinor fields on two different two-surfaces. Thus they represent a gauge ambiguity in the first and second order corrections, and the physical quantities must be invariant with respect to the substitutions $\lambda_{\underline{A}}{}^{(1)}\mapsto\lambda_{\underline{A}}{}^{(1)}+C_{\mathbf{A}}\lambda^{\mathbf{A}\,(0)}_{\underline{A}}$ and $\lambda_{\underline{A}}{}^{(2)}\mapsto\lambda_{\underline{A}}{}^{(2)}+D_{\mathbf{A}}\lambda^{\mathbf{A}\,(0)}_{\underline{A}}$ for any constants $C_{\mathbf{A}}$ and $D_{\mathbf{A}}$.

To compute the spin-angular momentum based on equation (2.5), first observe that the integral of the r^2 order term is vanishing, because that is just the spin-angular momentum in Minkowski spacetime. However, the r order term of the integrand is

$$
\begin{aligned}
&\lambda^{\mathbf{A}(0)}_0\lambda^{\mathbf{B}(1)}_1+\lambda^{\mathbf{A}(1)}_0\lambda^{\mathbf{B}(0)}_1+\lambda^{\mathbf{A}(0)}_1\lambda^{\mathbf{B}(1)}_0+\lambda^{\mathbf{A}(1)}_1\lambda^{\mathbf{B}(0)}_0=\\
&=-2\,{}_0\eth\big(\lambda^{\mathbf{A}(0)}_1\lambda^{\mathbf{B}(1)}_1+\lambda^{\mathbf{B}(0)}_1\lambda^{\mathbf{A}(1)}_1\big)+2\lambda^{\mathbf{A}(0)}_1\big({}_0\eth\lambda^{\mathbf{B}(1)}_1+\frac{1}{2}\lambda^{\mathbf{B}(1)}_0\big)+2\lambda^{\mathbf{B}(0)}_1\big({}_0\eth\lambda^{\mathbf{A}(1)}_1+\frac{1}{2}\lambda^{\mathbf{A}(1)}_0\big)=\\
&=-2\,{}_0\eth\big(\lambda^{\mathbf{A}(0)}_1\lambda^{\mathbf{B}(1)}_1+\lambda^{\mathbf{B}(0)}_1\lambda^{\mathbf{A}(1)}_1\big)-2\Big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\Big)\big(\lambda^{\mathbf{A}(0)}_1\lambda^{\mathbf{B}(0)}_0+\lambda^{\mathbf{B}(0)}_1\lambda^{\mathbf{A}(0)}_0\big),
\end{aligned}
$$

where we used equations (4.1) and (4.2); furthermore, by (4.1) the last of the second term is a total divergence:

$$
{}_0\eth^2\bar\sigma^0\big(\lambda^{\mathbf{A}(0)}_1\lambda^{\mathbf{B}(0)}_0+\lambda^{\mathbf{B}(0)}_1\lambda^{\mathbf{A}(0)}_0\big)={}_0\eth\Big({}_0\eth\bar\sigma^0\big(\lambda^{\mathbf{A}(0)}_1\lambda^{\mathbf{B}(0)}_0+\lambda^{\mathbf{B}(0)}_1\lambda^{\mathbf{A}(0)}_0\big)+\bar\sigma^0\lambda^{\mathbf{A}(0)}_0\lambda^{\mathbf{B}(0)}_0\Big). \tag{4.4}
$$

Thus for the anti-holomorphic spinor fields $J^{\mathbf{AB}}_r$ is diverging at $\mathcal{I}^+$ unless the integral

$$
L^{\mathbf{AB}}:=-\frac{1}{4\pi G}\oint_{\mathcal{S}}\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0\big)\Big(\lambda^{\mathbf{A}(0)}_1\lambda^{\mathbf{B}(0)}_0+\lambda^{\mathbf{A}(0)}_0\lambda^{\mathbf{B}(0)}_1\Big)\mathrm{d}\mathcal{S} \tag{4.5}
$$

is vanishing. (We return to the discussion of $L^{\mathbf{AB}}$ below.) Next let us consider the r^0 order term of (2.5):

$$
\begin{aligned}
&\frac{1}{2}\oint_{\mathcal{S}}\Bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(2)}+\lambda_0^{\mathbf{A}(1)}\lambda_1^{\mathbf{B}(1)}+\lambda_0^{\mathbf{A}(2)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(2)}+\lambda_1^{\mathbf{A}(1)}\lambda_0^{\mathbf{B}(1)}+\lambda_1^{\mathbf{A}(2)}\lambda_0^{\mathbf{B}(0)}-\\
&\qquad-\sigma^0\bar\sigma^0\bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\bigr)\Bigr)\mathrm{d}\mathcal{S}=\\
&=\oint_{\mathcal{S}}\Bigl(-\bigl({}_0\eth\lambda_1^{\mathbf{A}(0)}\bigr)\lambda_1^{\mathbf{B}(2)}+\lambda_1^{\mathbf{B}(1)}\bigl(-{}_0\eth\lambda_1^{\mathbf{A}(1)}-(\psi_0{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0)\lambda_0^{\mathbf{A}(0)}\bigr)+\frac{1}{2}\lambda_0^{\mathbf{A}(2)}\lambda_1^{\mathbf{B}(0)}-\\
&\qquad-\bigl({}_0\eth\lambda_1^{\mathbf{B}(0)}\bigr)\lambda_1^{\mathbf{A}(2)}+\lambda_1^{\mathbf{A}(1)}\bigl(-{}_0\eth\lambda_1^{\mathbf{B}(1)}-(\psi_0{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0)\lambda_0^{\mathbf{B}(0)}\bigr)+\frac{1}{2}\lambda_0^{\mathbf{B}(2)}\lambda_1^{\mathbf{A}(0)}-\\
&\qquad-\frac{1}{2}\sigma^0\bar\sigma^0\bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\bigr)\Bigr)\mathrm{d}\mathcal{S}=\\
&=\oint_{\mathcal{S}}\Bigl(\lambda_1^{\mathbf{A}(0)}\bigl({}_0\eth\lambda_1^{\mathbf{B}(2)}+\frac{1}{2}\lambda_0^{\mathbf{B}(2)}-\frac{1}{2}\sigma^0\bar\sigma^0\lambda_0^{\mathbf{B}(0)}\bigr)+\lambda_1^{\mathbf{B}(0)}\bigl({}_0\eth\lambda_1^{\mathbf{A}(2)}+\frac{1}{2}\lambda_0^{\mathbf{A}(2)}-\frac{1}{2}\sigma^0\bar\sigma^0\lambda_0^{\mathbf{A}(0)}\bigr)-\\
&\qquad-\bigl(\psi_0{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\bigr)\bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(1)}+\lambda_1^{\mathbf{A}(1)}\lambda_0^{\mathbf{B}(0)}\bigr)\Bigr)\mathrm{d}\mathcal{S}=\\
&=\oint_{\mathcal{S}}\Bigl(-\rho'^{(3)}\bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\bigr)+\\
&\qquad+\bigl(\psi_1{}^0+2\sigma^0\,{}_0\eth\bar\sigma^0\bigr)\lambda_1^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}+\sigma^0\bigl(\lambda_1^{\mathbf{A}(0)}\,{}_0\eth'\lambda_1^{\mathbf{B}(1)}+\lambda_1^{\mathbf{B}(0)}\,{}_0\eth'\lambda_1^{\mathbf{A}(1)}\bigr)-\\
&\qquad-\bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\bigr)\bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(1)}+\lambda_0^{\mathbf{A}(1)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(1)}+\lambda_1^{\mathbf{A}(1)}\lambda_0^{\mathbf{B}(0)}\bigr)\Bigr)\mathrm{d}\mathcal{S},
\end{aligned}
\tag{4.6}
$$

where $\rho'^{(3)}$ is the 3rd order term of ρ' in (3.7), and we used (4.1.a), (4.2.a) and (4.3.a). Substituting (3.7) here, using (4.1.b) and the consequences ${}_0\eth'\lambda_1^{\mathbf{A}(0)}=0$ and ${}_0\eth'\lambda_0^{\mathbf{A}(0)}=\lambda_1^{\mathbf{A}(0)}$ of (4.1), the integral of the first three terms of the right hand side of (4.6) can be written as

$$
\begin{aligned}
&\oint_{\mathcal{S}}\Bigl(\bigl(\psi_1{}^0+2\sigma^0\,{}_0\eth\bar\sigma^0\bigr)\lambda_1^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}-{}_0\eth'\sigma^0\bigl(\lambda_1^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(1)}+\lambda_1^{\mathbf{B}(0)}\lambda_1^{\mathbf{A}(1)}\bigr)+\bigl({}_0\eth\bar\sigma^0\,{}_0\eth'\sigma^0-\frac{1}{2}\sigma^0\bar\sigma^0+\\
&\qquad+\sigma^0\,{}_0\eth'{}_0\eth\bar\sigma^0+\bar\sigma^0\,{}_0\eth\,{}_0\eth'\sigma^0+\frac{1}{2}\,{}_0\eth'\psi_1{}^0+\frac{1}{2}\,{}_0\eth\bar\psi_{1'}{}^0-2G\phi_1{}^0\bar\phi_{1'}{}^0\bigr)\bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\bigr)\Bigr)\mathrm{d}\mathcal{S}=\\
&=\oint_{\mathcal{S}}\Bigl(\bigl(\frac{1}{2}\bar\psi_{1'}{}^0+\bar\sigma^0\,{}_0\eth'\sigma^0\bigr)\lambda_0^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}-{}_0\eth'\sigma^0\bigl(\lambda_1^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(1)}+\lambda_1^{\mathbf{B}(0)}\lambda_1^{\mathbf{A}(1)}\bigr)-\\
&\qquad-\bigl({}_0\eth'\sigma^0\,{}_0\eth\bar\sigma^0+\frac{1}{2}\sigma^0\bar\sigma^0+2G\phi_1{}^0\bar\phi_{1'}{}^0\bigr)\bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\bigr)\Bigr)\mathrm{d}\mathcal{S}=\\
&=\frac{1}{2}\oint_{\mathcal{S}}\Bigl(\bigl(\bar\psi_{1'}{}^0+2\bar\sigma^0\,{}_0\eth'\sigma^0+{}_0\eth'(\sigma^0\bar\sigma^0)\bigr)\lambda_0^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}-4G\,{}_0\eth'(\phi_1{}^0\bar\phi_{1'}{}^0)\bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\bigr)-\\
&\qquad-2\,{}_0\eth'\sigma^0\bigl(\lambda_1^{\mathbf{A}(0)}(\lambda_1^{\mathbf{B}(1)}+\lambda_0^{\mathbf{B}(0)}\,{}_0\eth\bar\sigma^0)+\lambda_1^{\mathbf{B}(0)}(\lambda_1^{\mathbf{A}(1)}+\lambda_0^{\mathbf{A}(0)}\,\eth\bar\sigma^0)\bigr)\Bigr)\mathrm{d}\mathcal{S}.
\end{aligned}
\tag{4.7}
$$

Using (4.2.a) again, the integral of the last term on the right hand side of (4.6) can be rewritten as

$$
\begin{aligned}
&-\oint_{\mathcal{S}}\bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\bigr)\Bigl(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(1)}+\lambda_0^{\mathbf{A}(1)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(1)}+\lambda_1^{\mathbf{A}(1)}\lambda_0^{\mathbf{B}(0)}\Bigr)\mathrm{d}\mathcal{S}=\\
&=\oint_{\mathcal{S}}\Bigl(\lambda_1^{\mathbf{B}(1)}\bigl({}_0\eth\lambda_1^{\mathbf{A}(1)}+\frac{1}{2}\lambda_0^{\mathbf{A}(1)}\bigr)+\lambda_1^{\mathbf{A}(1)}\bigl({}_0\eth\lambda_1^{\mathbf{B}(1)}+\frac{1}{2}\lambda_0^{\mathbf{B}(1)}\bigr)-\\
&\qquad-\bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\bigr)\bigl(\lambda_0^{\mathbf{A}(1)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(1)}\bigr)\Bigr)\mathrm{d}\mathcal{S}=\\
&=\oint_{\mathcal{S}}\Bigl\{\lambda_0^{\mathbf{A}(1)}\Bigl(\frac{1}{2}\lambda_1^{\mathbf{B}(1)}-\bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\bigr)\lambda_1^{\mathbf{B}(0)}\Bigr)+\\
&\qquad+\lambda_0^{\mathbf{B}(1)}\Bigl(\frac{1}{2}\lambda_1^{\mathbf{A}(1)}-\bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\bigr)\lambda_1^{\mathbf{A}(0)}\Bigr)\Bigr\}\mathrm{d}\mathcal{S}.
\end{aligned}
\tag{4.8}
$$

Therefore, $\lambda_0^{\mathbf{A}(2)}$ and $\lambda_1^{\mathbf{A}(2)}$ are not needed to calculate the anti-holomorphic spin-angular momentum, but $\lambda_0^{\mathbf{A}(1)}$ and $\lambda_1^{\mathbf{A}(1)}$ do appear in (4.6) explicitly. Since, however, the physical quantities should not be sensitive to the addition of the spurious zeroth order solutions to $\lambda_0^{\mathbf{A}(1)}$ and $\lambda_1^{\mathbf{A}(1)}$, and by such solutions $\lambda_0^{\mathbf{A}(1)}=0$ can always be achieved, whenever (4.8) gives zero, that should be a gauge term. To see that, in fact, this is the case, recall that (4.1.b) and (4.2.b) are the same, thus we may write $\lambda_0^{\mathbf{A}(1)}=C^{\mathbf{A}}{}_{\mathbf{C}}\lambda_0^{\mathbf{C}(0)}=-2C^{\mathbf{A}}{}_{\mathbf{C}}\,{}_0\eth\lambda_1^{\mathbf{C}(0)}$ for some constant complex 2×2 matrix $C^{\mathbf{A}}{}_{\mathbf{C}}$. (However, $C^{\mathbf{A}}{}_{\mathbf{B}}$ is not quite arbitrary, that is restricted by the requirement that the pair of anti-holomorphic spinor fields should form a normalized spin frame. In fact, from $\varepsilon^{\mathbf{AB}}=\varepsilon^{\underline{A}\underline{B}}(\lambda_{\underline{A}}^{\mathbf{A}(0)}+\frac{1}{r}\lambda_{\underline{A}}^{\mathbf{A}(1)}+...)(\lambda_{\underline{B}}^{\mathbf{B}(0)}+\frac{1}{r}\lambda_{\underline{B}}^{\mathbf{B}(1)}+...)$ it follows that $C^{\mathbf{AB}}$ must be symmetric.) Substituting this into (4.8) and using (4.2.a) we obtain

$$
\begin{aligned}
-2\big(&C^{\mathbf{A}}{}_{\mathbf{C}}\delta^{\mathbf{B}}_{\mathbf{D}}+C^{\mathbf{B}}{}_{\mathbf{C}}\delta^{\mathbf{A}}_{\mathbf{D}}\big)\oint_{\mathcal{S}}{}_0\eth\lambda_1^{\mathbf{C}(0)}\Big(\frac{1}{2}\lambda_1^{\mathbf{D}(1)}-\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\big)\lambda_1^{\mathbf{D}(0)}\Big)\mathrm{d}\mathcal{S}=\\
&=\big(C^{\mathbf{A}}{}_{\mathbf{C}}\delta^{\mathbf{B}}_{\mathbf{D}}+C^{\mathbf{B}}{}_{\mathbf{C}}\delta^{\mathbf{A}}_{\mathbf{D}}\big)\oint_{\mathcal{S}}\lambda_1^{\mathbf{C}(0)}\Big({}_0\eth\lambda_1^{\mathbf{D}(1)}-2\,{}_0\eth\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\big)\lambda_1^{\mathbf{D}(0)}+\\
&\qquad\qquad+\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\big)\lambda_0^{\mathbf{D}(0)}\Big)\mathrm{d}\mathcal{S}=\\
&=\big(C^{\mathbf{A}}{}_{\mathbf{C}}\delta^{\mathbf{B}}_{\mathbf{D}}+C^{\mathbf{B}}{}_{\mathbf{C}}\delta^{\mathbf{A}}_{\mathbf{D}}\big)\oint_{\mathcal{S}}\lambda_1^{\mathbf{C}(0)}\Big(-\frac{1}{2}\lambda_0^{\mathbf{D}(1)}-2\,{}_0\eth\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\big)\lambda_1^{\mathbf{D}(0)}\Big)\mathrm{d}\mathcal{S}=\\
&=\frac{1}{4}\big(C^{\mathbf{A}}{}_{\mathbf{C}}C^{\mathbf{B}}{}_{\mathbf{D}}+C^{\mathbf{B}}{}_{\mathbf{C}}C^{\mathbf{A}}{}_{\mathbf{D}}\big)\oint_{\mathcal{S}}\lambda_0^{\mathbf{C}(0)}\lambda_1^{\mathbf{D}(0)}\mathrm{d}\mathcal{S}+\\
&-\big(C^{\mathbf{A}}{}_{\mathbf{C}}\delta^{\mathbf{B}}_{\mathbf{D}}+C^{\mathbf{B}}{}_{\mathbf{C}}\delta^{\mathbf{A}}_{\mathbf{D}}\big)\oint_{\mathcal{S}}\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\big)\Big(\lambda_0{}^{\mathbf{C}(0)}\lambda_1{}^{\mathbf{D}(0)}+\lambda_1{}^{\mathbf{C}(0)}\lambda_0{}^{\mathbf{D}(0)}\Big)\mathrm{d}\mathcal{S}.
\end{aligned}
\tag{4.9}
$$

However, the first integral on the right hand side of (4.9) is vanishing, while, taking into account (4.4), the second is seen to be proportional to $L^{\mathbf{AB}}$ above. Therefore, the anti-holomorphic spin-angular momentum for the large sphere of radius r is

$$
J_r^{\mathbf{AB}}=rL^{\mathbf{AB}}+\big(C^{\mathbf{A}}{}_{\mathbf{C}}\delta^{\mathbf{B}}_{\mathbf{D}}+C^{\mathbf{B}}{}_{\mathbf{C}}\delta^{\mathbf{A}}_{\mathbf{D}}\big)L^{\mathbf{CD}}-E^{\mathbf{AB}}+I^{\mathbf{AB}}+O(r^{-1}),
\tag{4.10}
$$

where we introduced the notations

$$
E^{\mathbf{AB}}:=\frac{1}{2\pi}\oint_{\mathcal{S}}\phi_1{}^0\bar\phi_{1'}{}^0\Big(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}+\lambda_1^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)}\Big)\mathrm{d}\mathcal{S},
\tag{4.11}
$$

$$
\begin{aligned}
I^{\mathbf{AB}}:=&\frac{1}{8\pi G}\oint_{\mathcal{S}}\Big\{\Big(\bar\psi_{1'}{}^0+2\bar\sigma^0\,{}_0\eth'\sigma^0+{}_0\eth'\big(\sigma^0\bar\sigma^0\big)\Big)\lambda_0^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}-\\
&-2\,{}_0\eth'\sigma^0\Big(\lambda_1^{\mathbf{A}(0)}\big(\lambda_1^{\mathbf{B}(1)}+\lambda_0^{\mathbf{B}(0)}\,{}_0\eth\bar\sigma^0\big)+\lambda_1^{\mathbf{B}(0)}\big(\lambda_1^{\mathbf{A}(1)}+\lambda_0^{\mathbf{A}(0)}\,{}_0\eth\bar\sigma^0\big)\Big)\Big\}\mathrm{d}\mathcal{S}.
\end{aligned}
\tag{4.12}
$$

Therefore, as we noted above, the quasi-local spin-angular momentum based on Bramson's superpotential and the anti-holomorphic spinors is diverging at the future null infinity, furthermore, its finite part is ambiguous unless $L^{\mathbf{AB}}$ is vanishing. In addition, contrary to expectations, the electromagnetic field contributes to $J_r^{\mathbf{AB}}$ in the $O(1)$ order. On the other hand, the first three terms together in the integrand of $I^{\mathbf{AB}}$ is just the integrand of Bramson's specific spin-angular momentum expression based on the asymptotic twistor equation [22]. Though in the present case the spinor fields $\lambda_A^{\mathbf{A}}$ are anti-holomorphic and not the solutions of the asymptotic twistor equations, the zeroth order part of $\lambda_A^{\mathbf{A}}$, which appears as the coefficient of the first three terms of the integrand of $I^{\mathbf{AB}}$, coincides with the zeroth order part of the solutions of the asymptotic twistor equations. Furthermore, the coefficient $\lambda_0^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}$ is just minus the generator of the anti-self-dual rotations on the $u=0$ cut (the 'origin') of $\mathcal{I}^+$ (see Section 3. and the Appendix). It might be worth

emphasizing that these generators of the anti-self-dual rotation BMS vector fields emerged naturally, like the approximate rotation-boost Killing vectors in the small sphere calculations [37], without putting them into the general formulae by hand. Although $I^{\mathbf{AB}}$ depends on the solution $\lambda_1^{\mathbf{A}(1)}$ of (4.2.a) (and hence its integrand is a genuinely non-local expression), that is independent of the gauge solutions. In fact, using ${}_0\eth'\lambda_1^{\mathbf{A}(0)}=0$, it is easy to see that the addition of a gauge solution to $\lambda_1^{\mathbf{A}(1)}$ changes the integrand by a total ${}_0\eth'$-derivative. To clarify the meaning of the term of the integrand involving $\lambda_1^{\mathbf{A}(1)}$, recall that the BMS vector fields $k^a_{\mathbf{AB}}$ are tangent only to the origin cut, and they can be represented completely by $\lambda_0^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}$ only there. On general cuts $k^a_{\mathbf{AB}}$ contain correction terms proportional to the generator $(\frac{\partial}{\partial u})^a$ of $\mathcal{I}^+$. Then the integral of the term involving $\lambda_1^{\mathbf{A}(1)}$ can be considered as the contribution of this correction term built in a non-local way from the fields on the actual cut and the spinor constituents of the BMS rotations on the origin cut.

However, $L^{\mathbf{AB}}$ is not zero in general, because its components are proportional to that of the spatial part of the Bondi–Sachs energy-momentum

$$ {}_\infty P^{\mathbf{AB}'}:=-\frac{1}{4\pi G}\oint_{\mathcal{S}}\bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0\bigr)\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\mathrm{d}\mathcal{S}. \tag{4.13}$$

To see this, let us substitute the explicit solutions (A.2.2) into (4.5). We obtain $L^{\mathbf{00}}=-\sqrt{2}\,{}_\infty P^{\mathbf{01}'}$, $L^{\mathbf{01}}=\frac{1}{\sqrt2}({}_\infty P^{\mathbf{00}'}-{}_\infty P^{\mathbf{11}'})$ and $L^{\mathbf{11}}=\sqrt{2}\,{}_\infty P^{\mathbf{10}'}$, i.e. $L^{\mathbf{AB}}$ represents the linear momentum. In fact, for $\mathbf{A}=\mathbf{B}=0$, $\mathbf{A}=\mathbf{B}=1$ and $\mathbf{A}=0$, $\mathbf{B}=1$ the coefficient of $\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0$ in (4.5) is $-2\zeta(1+\zeta\bar\zeta)^{-1}$, $2\bar\zeta(1+\zeta\bar\zeta)^{-1}$ and $(\zeta\bar\zeta-1)(1+\zeta\bar\zeta)^{-1}$, respectively, which are proportional to the independent spatial BMS translations. Therefore, the anti-holomorphic spin-angular momentum can be finite only in the center-of-mass system (i.e. when the spatial components of the Bondi–Sachs energy-momentum are vanishing), and hence $I^{\mathbf{AB}}$ in the $O(1)$ part of (4.10) appears to represent only the intrinsic or *spin part* of the total angular momentum, while $rL^{\mathbf{AB}}$ appears to be the orbital part of the angular momentum. To check whether this interpretation is correct we should calculate the quasi-local Pauli–Lubanski spin (2.4) built from the quasi-local anti-holomorphic Dougan–Mason energy-momentum $P^{\mathbf{AB}'}$ and the anti-holomorphic spin-angular momentum. However, to compute the spin, we need to know the Dougan–Mason energy-momentum for large spheres with $O(r^{-1})$ accuracy. Since this has been calculated only in stationary spacetimes [39] (where a physical term was apparently overlooked and the gauge ambiguity caused by the spurious solutions was not considered at all), first we must calculate this.

The r order part of (2.6) is vanishing, because $\oint_{\mathcal{S}}\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\mathrm{d}\mathcal{S}=2\oint_{\mathcal{S}}\lambda_1^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(0)}\mathrm{d}\mathcal{S}=4\pi\,\sigma_0^{\mathbf{AB}'}$ (see Appendix A.2), i.e. $P_r^{\mathbf{AB}'}$ has a finite $r\to\infty$ limit at $\mathcal{I}^+$. Substituting (3.5) and (3.7) into the finite term of (2.6), using (4.1.a), ${}_0\eth'\lambda_0^{\mathbf{A}(0)}=\lambda_1^{\mathbf{A}(0)}$ and its complex conjugate, (4.2.a) and the fact that $\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0$ is real, we obtain

$$\begin{aligned}
&\oint_{\mathcal{S}}\Bigl\{\frac12\Bigl(\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(1)}+\lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\Bigr)-\Bigl(\lambda_1^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(1)}+\lambda_1^{\mathbf{A}(1)}\bar\lambda_{1'}^{\mathbf{B}'(0)}\Bigr)+\\
&\qquad+\Bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\Bigr)\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\Bigr\}\mathrm{d}\mathcal{S}=\\
&=\oint_{\mathcal{S}}\Bigl\{\bar\lambda_{0'}^{\mathbf{B}'(0)}\Bigl({}_0\eth\lambda_1^{\mathbf{A}(1)}+\frac12\lambda_0^{\mathbf{A}(1)}\Bigr)+\lambda_0^{\mathbf{A}(0)}\Bigl({}_0\eth'\bar\lambda_{1'}^{\mathbf{B}'(1)}+\frac12\bar\lambda_{0'}^{\mathbf{B}'(1)}\Bigr)+\\
&\qquad+\Bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\Bigr)\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\Bigr\}\mathrm{d}\mathcal{S}=\\
&=-\oint_{\mathcal{S}}\Bigl(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\Bigr)\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\mathrm{d}\mathcal{S}.
\end{aligned}$$

Since the last term of the integrand is a total ${}_0\eth$-derivative: $({}_0\eth^2\bar\sigma^0)\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}={}_0\eth({}_0\eth\bar\sigma^0\,\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)})-({}_0\eth\bar\sigma^0)\lambda_0^{\mathbf{A}(0)}\,{}_0\eth\bar\lambda_{0'}^{\mathbf{B}'(0)}={}_0\eth({}_0\eth\bar\sigma^0\,\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}-\bar\sigma^0\lambda_0^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(0)})$, the finite part of (2.6) is, in fact, the Bondi–Sachs energy-momentum (4.13).

The $O(r^{-1})$ term of (2.6) is

$$
\begin{aligned}
&\oint_{\mathcal S}\Big(\big(\rho'^{(3)}-\frac{1}{2}\sigma^0\bar\sigma^0\big)\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}+\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\big)\big(\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(1)}+\lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\big)+\\
&\quad+\frac{1}{2}\big(\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(2)}+\lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(1)}+\lambda_0^{\mathbf{A}(2)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\big)-{}_0\eth'\lambda_0^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(2)}-\lambda_1^{\mathbf{A}(2)}\,{}_0\eth\bar\lambda_{0'}^{\mathbf{B}'(0)}-\lambda_1^{\mathbf{A}(1)}\bar\lambda_{1'}^{\mathbf{B}'(1)}\Big)\mathrm{d}\mathcal S=\\
&=\oint_{\mathcal S}\Big(\big(\rho'^{(3)}-\frac{1}{2}\sigma^0\bar\sigma^0\big)\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}+\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0\big)\big(\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(1)}+\lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\big)+\\
&\quad+\lambda_0^{\mathbf{A}(0)}\big({}_0\eth'\bar\lambda_{1'}^{\mathbf{B}'(2)}+\frac{1}{2}\bar\lambda_{0'}^{\mathbf{B}'(2)}\big)+\bar\lambda_{0'}^{\mathbf{B}'(0)}\big({}_0\eth\lambda_1^{\mathbf{A}(2)}+\frac{1}{2}\lambda_0^{\mathbf{A}(2)}\big)+\frac{1}{2}\lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(1)}-\lambda_1^{\mathbf{A}(1)}\bar\lambda_{1'}^{\mathbf{B}'(1)}\Big)\mathrm{d}\mathcal S=\\
&=\oint_{\mathcal S}\Big(-\big(2G\phi_1{}^0\bar\phi_{1'}{}^0+{}_0\eth\bar\sigma^0\,{}_0\eth'\sigma^0\big)\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}-\\
&\quad-\big({}_0\eth\bar\sigma^0\big)\lambda_0^{\mathbf{A}(0)}\bar\lambda_{1'}^{\mathbf{B}'(1)}-\big({}_0\eth'\sigma^0\big)\lambda_1^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(0)}+\frac{1}{2}\lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(1)}-\lambda_1^{\mathbf{A}(1)}\bar\lambda_{1'}^{\mathbf{B}'(1)}\Big)\mathrm{d}\mathcal S=\\
&=\oint_{\mathcal S}\Big(-2G\phi_1{}^0\bar\phi_{1'}{}^0\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}+\frac{1}{2}\lambda_0^{\mathbf{A}(1)}\bar\lambda_{0'}^{\mathbf{B}'(1)}-\big(\lambda_1^{\mathbf{A}(1)}+\lambda_0^{\mathbf{A}(0)}\,{}_0\eth\bar\sigma^0\big)\big(\bar\lambda_{1'}^{\mathbf{B}'(1)}+\bar\lambda_{0'}^{\mathbf{B}'(0)}\,{}_0\eth'\sigma^0\big)\Big)\mathrm{d}\mathcal S.
\end{aligned}
\tag{4.14}
$$

Here first we used ${}_0\eth'\lambda_0^{\mathbf{A}(0)}=\lambda_1^{\mathbf{A}(0)}$, and then (4.3.a), (4.1.b), (3.7) and the fact that $\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0+{}_0\eth^2\bar\sigma^0$ is real. However, as we saw above, $\lambda_0^{\mathbf{A}(1)}=C^{\mathbf{A}}{}_{\mathbf{C}}\lambda_0^{\mathbf{C}(0)}=-2C^{\mathbf{A}}{}_{\mathbf{C}}\,{}_0\eth\lambda_1^{\mathbf{C}(0)}$, and by means of which (4.2.a) takes the form ${}_0\eth(\lambda_1^{\mathbf{A}(1)}+\lambda_0^{\mathbf{A}(0)}\,{}_0\eth\bar\sigma^0)=C^{\mathbf{A}}{}_{\mathbf{C}}\,{}_0\eth\lambda_1^{\mathbf{C}(0)}-(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0)\lambda_0^{\mathbf{A}(0)}$. Since ${}_0\eth$ acting on $s=-\frac{1}{2}$ spin weight quantities is isomorphism (see e.g. [47,50]), we can write $\lambda_1^{\mathbf{A}(1)}+\lambda_0^{\mathbf{A}(0)}\,{}_0\eth\bar\sigma^0=C^{\mathbf{A}}{}_{\mathbf{C}}\lambda_1^{\mathbf{C}(0)}-{}_0\eth^{-1}((\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0)\lambda_0^{\mathbf{A}(0)})$. Substituting these into (4.12), using ${}_0\eth'\lambda_0^{\mathbf{A}(0)}=\lambda_1^{\mathbf{A}(0)}$, (4.13) and the expression for the L_2-scalar product of the components $\lambda_0^{\mathbf{A}(0)}$ and $\lambda_1^{\mathbf{A}(0)}$ given in Appendix A.2, finally we obtain

$$
P_r^{\mathbf{A}\mathbf{B}'}={}_\infty P^{\mathbf{A}\mathbf{B}'}+\frac{1}{r}\big(C^{\mathbf{A}}{}_{\mathbf{C}}\delta^{\mathbf{B}'}_{\mathbf{D}'}+\bar C^{\mathbf{B}'}{}_{\mathbf{D}'}\delta^{\mathbf{A}}_{\mathbf{C}}\big){}_\infty P^{\mathbf{C}\mathbf{D}'}-\frac{1}{r}F^{\mathbf{A}\mathbf{B}'}+\frac{1}{r}M^{\mathbf{A}\mathbf{B}'}+O(\frac{1}{r^2}),\tag{4.15}
$$

where we used the notations

$$
F^{\mathbf{A}\mathbf{B}'}:=\frac{1}{2\pi}\oint_{\mathcal S}\phi_1{}^0\bar\phi_{1'}{}^0\lambda_0^{\mathbf{A}(0)}\bar\lambda_{0'}^{\mathbf{B}'(0)}\mathrm{d}\mathcal S,\tag{4.16}
$$

$$
M^{\mathbf{A}\mathbf{B}'}:=-\frac{1}{4\pi G}\oint_{\mathcal S}{}_0\eth^{-1}\Big(\big(\psi_2{}^0+\sigma^0\dot{\bar\sigma}^0\big)\lambda_0^{\mathbf{A}(0)}\Big)\,{}_0\eth'^{-1}\Big(\big(\bar\psi_{2'}{}^0+\bar\sigma^0\dot\sigma^0\big)\bar\lambda_{0'}^{\mathbf{B}'(0)}\Big)\mathrm{d}\mathcal S.\tag{4.16}
$$

Therefore, as it could be expected, the r^{-1} order term in the expansion of the anti-holomorphic Dougan–Mason energy-momentum is ambiguous, and, in addition to the electromagnetic contribution, there is an extra term. Note that $E^{\mathbf{A}\mathbf{B}}$ above is related to $F^{\mathbf{A}\mathbf{B}'}$ like $L^{\mathbf{A}\mathbf{B}}$ to ${}_\infty P^{\mathbf{A}\mathbf{B}'}$, and hence $E^{\mathbf{00}}=-\sqrt{2}F^{\mathbf{01}'}$, $E^{\mathbf{01}}=\frac{1}{\sqrt{2}}(F^{\mathbf{00}'}-F^{\mathbf{11}'})$ and $E^{\mathbf{11}}=\sqrt{2}F^{\mathbf{10}'}$.

Substituting (4.10) and (4.15) into (2.4) and using how the components of $L^{\mathbf{A}\mathbf{B}}$ and $E^{\mathbf{A}\mathbf{B}}$ are related to those of ${}_\infty P^{\mathbf{A}\mathbf{B}'}$ and $F^{\mathbf{A}\mathbf{B}'}$, respectively, we obtain

$$
S_{\mathbf{A}\mathbf{B}'}=\mathrm{i}\Big({}_\infty P_{\mathbf{A}\mathbf{C}'}\bar I^{\mathbf{C}'}{}_{\mathbf{B}'}+M_{\mathbf{A}\mathbf{C}'}\bar L^{\mathbf{C}'}{}_{\mathbf{B}'}-{}_\infty P_{\mathbf{B}'\mathbf{C}}I^{\mathbf{C}}{}_{\mathbf{A}}-M_{\mathbf{B}'\mathbf{C}}L^{\mathbf{C}}{}_{\mathbf{A}}\Big).\tag{4.17}
$$

The diverging term, the ambiguities and the contribution of the electromagnetic field disappeared. Therefore, *the quasi-local Pauli–Lubanski spin vector built from the anti-holomorphic Dougan–Mason energy-momentum and the anti-holomorphic spin-angular momentum (2.2) has finite limit at the future null infinity, it is free*

of ambiguities, and is built only from the gravitational data. Note that our Pauli–Lubanski spin is free of the so-called supertranslation ambiguity, because this is defined in terms of the solutions of an elliptic differential equation on the cut in question, and not by means of the BMS boost-rotation vector fields. Thus the present construction is similar in its spirit to that of Penrose [12]. The fact that we could derive only a Pauli–Lubanski spin is compatible with the idea of Bergmann and Thomson [21] that the gravitational angular momentum should be analogous to spin (justifying the 'spin-angular momentum' terminology), but raises the question as whether that should be completed by an orbital angular momentum part or not. Another interesting issue is how the Pauli–Lubanski spin changes under (infinitesimal) supertranslations, in particular, under time translations; or whether there exists a flux for $S_{\mathbf{AA}'}$ through $\mathcal{I}^+$ or not. However, these questions are beyond the scope of the present paper.

5. Stationary spacetimes

Suppose that the spacetime is stationary. First we show that all the terms of the integrand of (4.12) involving the asymptotic shear together integrates to zero. Bramson already showed that $2\bar\sigma^0\,{}_0\eth'\sigma^0 + {}_0\eth'(\sigma^0\bar\sigma^0) = {}_0\eth'^3(S\,{}_0\eth^2S + \frac{1}{2}({}_0\eth S)^2) - {}_0\eth(S\,{}_0\eth'^3\,{}_0\eth S + 3\,{}_0\eth'^2\,{}_0\eth S\,{}_0\eth S + 3({}_0\eth' S)^2)$, which, together with ${}_0\eth\lambda_0^{\mathbf{A}(0)} = 0$, ${}_0\eth'\lambda_0^{\mathbf{A}(0)} = \lambda_1^{\mathbf{A}(0)}$ and ${}_0\eth'\lambda_1^{\mathbf{A}(0)} = 0$, gives that the first two σ^0-terms of the integrand of (4.12) give zero. To evaluate the last term of the integrand, first we must solve (4.2.a). In stationary spacetimes (4.2.a) takes the form ${}_0\eth\lambda_1^{\mathbf{A}(1)} + \frac{1}{2}C^{\mathbf{A}}{}_{\mathbf{B}}\lambda_0^{\mathbf{B}(0)} = (GM + {}_0\eth^2\,{}_0\eth'^2S)\lambda_0^{\mathbf{A}(0)}$. Using $\dim\ker\eth_{(-1,0)} = 0$, this can be solved explicitly. Its solution is $\lambda_1^{\mathbf{A}(1)} = {}_0\eth({}_0\eth'^2S)\lambda_0^{\mathbf{A}(0)} + (-2GM\delta^{\mathbf{A}}_{\mathbf{B}} + C^{\mathbf{A}}{}_{\mathbf{B}})\lambda_1^{\mathbf{B}(0)}$. Substituting this into the last term of the integrand, that takes the form

$$
\begin{aligned}
&-2\,{}_0\eth'\sigma^0\Big(\lambda_1^{\mathbf{A}(0)}\big(\lambda_1^{\mathbf{B}(1)} + \lambda_0^{\mathbf{B}(0)}\,{}_0\eth\bar\sigma^0\big) + \lambda_1^{\mathbf{B}(0)}\big(\lambda_1^{\mathbf{A}(1)} + \lambda_0^{\mathbf{A}(0)}\,{}_0\eth\bar\sigma^0\big)\Big) = \\
&= -2\Big(\big(-2GM\delta^{\mathbf{A}}_{\mathbf{C}} + C^{\mathbf{A}}{}_{\mathbf{C}}\big)\delta^{\mathbf{B}}_{\mathbf{D}} + \big(-2GM\delta^{\mathbf{B}}_{\mathbf{D}} + C^{\mathbf{B}}{}_{\mathbf{D}}\big)\delta^{\mathbf{A}}_{\mathbf{C}}\Big)\,{}_0\eth'\big(\sigma^0\lambda_1^{\mathbf{C}(0)}\lambda_1^{\mathbf{D}(0)}\big).
\end{aligned}
$$

Therefore, the last term of the integrand of (4.12) integrates to zero, too. By ${\psi_2}^0 = -Gm$ the integrand of $L^{\mathbf{AB}}$ is the total ${}_0\eth'$-derivative ${}_0\eth'(-Gm\lambda_0^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)})$, and by ${\phi_1}^0 = \frac{1}{2}(e + \mathrm{i}\mu)$ the integrand of $E^{\mathbf{AB}}$ is ${}_0\eth'(\frac{1}{4}[e^2+\mu^2]\lambda_0^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)})$. Thus in stationary spacetimes (4.10) reduces to

$$
J_r^{\mathbf{AB}} = \frac{1}{8\pi G}\oint_{\mathcal{S}}\bar\psi_{1'}{}^0\lambda_0^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\mathrm{d}\mathcal{S} + O(r^{-1}), \tag{5.1}
$$

which is the 'standard' expression for the angular momentum in stationary spacetimes. In fact, substituting ${\psi_1}^0 = -3G\,{}_0\eth(MS + \mathrm{i}\sum_{m=-1}^{m=1}J_mY_{1,m})$ here and using ${}_0\eth'\lambda_0^{\mathbf{A}(0)} = \lambda_1^{\mathbf{A}(0)}$, we find

$$
J_r^{\mathbf{AB}} = \frac{3M}{8\pi}\oint_{\mathcal{S}}S\Big(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)} + \lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\Big)\mathrm{d}\mathcal{S} - \frac{3\mathrm{i}}{8\pi}\sum_{m=-1}^{1}\overline{J_m}\oint_{\mathcal{S}}\overline{Y_{1,m}}\Big(\lambda_0^{\mathbf{A}(0)}\lambda_1^{\mathbf{B}(0)} + \lambda_1^{\mathbf{A}(0)}\lambda_0^{\mathbf{B}(0)}\Big)\mathrm{d}\mathcal{S} + O(r^{-1}).
$$

Substituting the specific solutions (A.2.2) here and defining $c_m := M\oint_{\mathcal{S}}\overline{Y_{1,m}}S\mathrm{d}\mathcal{S}$ and the corresponding real components $c^{\mathbf{i}} = (c^1, c^2, c^3)$ by $\sum_{m=-1}^{m=1}c_mY_{1,m} =: c^1\sin\theta\cos\phi + c^2\sin\theta\sin\phi + c^3\cos\theta$, for the leading terms we finally obtain $J^{\mathbf{ij}} = \varepsilon^{\mathbf{ijk}}j_{\mathbf{k}}$ and $J^{\mathbf{0k}} = -c^{\mathbf{k}}$. Thus $J^{\mathbf{AB}}$ reproduces the $j_{\mathbf{k}}$'s, and the $c^{\mathbf{k}}$'s can be interpreted as the components of the relativistic center-of-mass. In particular, for the Kerr–Newman solution on the shear-free cuts $J^{\mathbf{12}} = -Ma$. Using the expressions of Appendix A.2 for the L_2 scalar product of the spinor components $\lambda_{\underline{A}}^{\mathbf{A}(0)}$, we obtain ${}_\infty P^{\mathbf{AB}'} = M\,\sigma_0^{\mathbf{AB}'}$, $F^{\mathbf{AB}'} = \frac{1}{2}(e^2+\mu^2)\,\sigma_0^{\mathbf{AB}'}$ and $M^{\mathbf{AB}'} = -2GM^2\,\sigma_0^{\mathbf{AB}'}$. Hence in stationary spacetimes the Pauli–Lubanski spin reduces to that of Bramson. As he showed [23], this is invariant with respect to supertranslations.

6. The holomorphic spin-angular momentum

For the components $\lambda_{\underline{A}} =: \lambda_{\underline{A}}{}^{(0)} + \frac{1}{r}\lambda_{\underline{A}}{}^{(1)} + \frac{1}{r^2}\lambda_{\underline{A}}{}^{(2)} + ...$ of the holomorphic spinor fields on $\mathcal{S}_{(u,r)}$ we obtain the equations

$$
\begin{aligned}
&{}_0\eth'\lambda_0{}^{(0)} - \lambda_1{}^{(0)} = 0, && (6.1.a)\\
&{}_0\eth'\lambda_1{}^{(0)} - \dot{\bar\sigma}^0\lambda_0{}^{(0)} = 0, && (6.1.b)\\
&{}_0\eth'\lambda_0{}^{(1)} - \lambda_1{}^{(1)} = {}_0\eth\big(\bar\sigma^0\lambda_0{}^{(0)}\big), && (6.2.a)\\
&{}_0\eth'\lambda_1{}^{(1)} - \dot{\bar\sigma}^0\lambda_0{}^{(1)} = \bar\sigma^0\Big({}_0\eth\lambda_1{}^{(0)} + \frac{1}{2}\lambda_0{}^{(0)}\Big) - {}_0\eth'\Big(\big({}_0\eth\bar\sigma^0\big)\lambda_0{}^{(0)}\Big), && (6.2.b)\\
&{}_0\eth'\lambda_0{}^{(2)} - \lambda_1{}^{(2)} = {}_0\eth\big(\bar\sigma^0\lambda_0{}^{(1)}\big) - \big(\frac{1}{2}\bar\psi_{1'}{}^0 + \bar\sigma^0\,{}_0\eth'\sigma^0\big)\lambda_0{}^{(0)}, && (6.3.a)\\
&{}_0\eth'\lambda_1{}^{(2)} - \dot{\bar\sigma}^0\lambda_0{}^{(2)} = \bar\sigma^0\Big({}_0\eth\lambda_1{}^{(1)} + \frac{1}{2}\lambda_0{}^{(1)}\Big) - {}_0\eth'\Big(\big({}_0\eth\bar\sigma^0\big)\lambda_0{}^{(1)}\Big) + \Big(\frac{1}{2}\bar\psi_{1'}{}^0 + \bar\sigma^0\,{}_0\eth'\sigma^0\Big)\lambda_1{}^{(0)} + \\
&\quad + \bar\sigma^0\,{}_0\eth\bar\sigma^0\,{}_0\eth\lambda_0{}^{(0)} + \Big(\frac{1}{2}\bar\sigma^0\psi_2{}^0 + \frac{1}{2}\,{}_0\eth'\bar\psi_{1'}{}^0 + {}_0\eth'\big(\bar\sigma^0\,{}_0\eth\sigma^0\big) + \bar\sigma^0\,{}_0\eth^2\bar\sigma^0 - G\phi_2{}^0\bar\phi_{0'}{}^0\Big)\lambda_0{}^{(0)}. && (6.3.b)
\end{aligned}
$$

Therefore, the zeroth order holomorphic spinors are *not* constant unless $\dot\sigma^0$ is vanishing, and hence the coefficient of the r^2 order term of (2.5) is not zero in general. Therefore, *the quasi-local spin-angular momentum (2.2) based the holomorphic spinor fields is diverging near $\mathcal{I}^+$ in presence of outgoing gravitational radiation.* Thus let us concentrate on spacetimes for which $\dot\sigma^0 = 0$. Then $\lambda_0{}^{(0)}$, $\lambda_1{}^{(0)}$ are components of a constant spinor field on $\mathcal{S}$, implying the vanishing of the r^2 order term in (2.5), and the vanishing of the first term on the right hand side of (6.2.b). Then, however, its general solution is $\lambda_1^{\mathbf{A}\,(1)} = -({}_0\eth\bar\sigma^0)\lambda_0^{\mathbf{A}\,(0)} + C^{\mathbf{A}}{}_{\mathbf{B}}\lambda_1^{\mathbf{B}\,(0)}$ for some 2×2 complex matrix $C^{\mathbf{A}}{}_{\mathbf{B}}$, by means of which the solution of (6.2.a) is $\lambda_0^{\mathbf{A}\,(1)} = C^{\mathbf{A}}{}_{\mathbf{B}}\lambda_0^{\mathbf{B}\,(0)}$. Therefore, the integrand of the r order term of (2.5) is

$$
\begin{aligned}
&\lambda_0^{\mathbf{A}\,(0)}\lambda_1^{\mathbf{B}\,(1)} + \lambda_0^{\mathbf{A}\,(1)}\lambda_1^{\mathbf{B}\,(0)} + \lambda_1^{\mathbf{A}\,(0)}\lambda_0^{\mathbf{B}\,(1)} + \lambda_1^{\mathbf{A}\,(1)}\lambda_0^{\mathbf{B}\,(0)} = \\
&= \big(\delta^{\mathbf{A}}_{\mathbf{C}}C^{\mathbf{B}}{}_{\mathbf{D}} + C^{\mathbf{A}}{}_{\mathbf{C}}\delta^{\mathbf{B}}_{\mathbf{D}}\big)\big(\lambda_0^{\mathbf{C}\,(0)}\lambda_1^{\mathbf{D}\,(0)} + \lambda_0^{\mathbf{D}\,(0)}\lambda_1^{\mathbf{C}\,(0)}\big) - 2\,{}_0\eth\big(\bar\sigma^0\lambda_0^{\mathbf{A}\,(0)}\lambda_0^{\mathbf{B}\,(0)}\big).
\end{aligned}
$$

However, its integral is zero, i.e. *in absence of outgoing gravitational radiation (i.e. if $\dot\sigma^0 = 0$) the quasi-local angular momentum based on (2.2) and the holomorphic spinor fields has a finite limit at the future null infinity.*

To calculate this finite value let us consider the r^0 order term of (2.5). Using the explicit solutions for $\lambda_0^{\mathbf{A}\,(1)}$ and $\lambda_1^{\mathbf{A}\,(1)}$ above, the fact that $\lambda_A^{\mathbf{A}\,(0)}$ is constant and (6.3.a) we obtain

$$
\begin{aligned}
&\oint_{\mathcal{S}}\Big(\lambda_0^{\mathbf{A}\,(0)}\lambda_1^{\mathbf{B}\,(2)} + \lambda_0^{\mathbf{A}\,(1)}\lambda_1^{\mathbf{B}\,(1)} + \lambda_0^{\mathbf{A}\,(2)}\lambda_1^{\mathbf{B}\,(0)} + \lambda_1^{\mathbf{A}\,(0)}\lambda_0^{\mathbf{B}\,(2)} + \lambda_1^{\mathbf{A}\,(1)}\lambda_0^{\mathbf{B}\,(1)} + \lambda_1^{\mathbf{A}\,(2)}\lambda_0^{\mathbf{B}\,(0)} - \\
&\qquad - \sigma^0\bar\sigma^0\big(\lambda_0^{\mathbf{A}\,(0)}\lambda_1^{\mathbf{B}\,(0)} + \lambda_1^{\mathbf{A}\,(0)}\lambda_0^{\mathbf{B}\,(0)}\big)\Big)\mathrm{d}\mathcal{S} = \\
&= \oint_{\mathcal{S}}\Big(\big({}_0\eth'\lambda_0^{\mathbf{B}\,(0)}\big)\lambda_0^{\mathbf{A}\,(2)} + \big({}_0\eth'\lambda_0^{\mathbf{A}\,(0)}\big)\lambda_0^{\mathbf{A}\,(2)} + \\
&\qquad + C^{\mathbf{A}}{}_{\mathbf{C}}\lambda_0^{\mathbf{C}\,(0)}\big(-{}_0\eth\bar\sigma^0\lambda_0^{\mathbf{B}\,(0)} + C^{\mathbf{B}}{}_{\mathbf{D}}\lambda_1^{\mathbf{D}\,(0)}\big) + C^{\mathbf{B}}{}_{\mathbf{D}}\lambda_0^{\mathbf{D}\,(0)}\big(-{}_0\eth\bar\sigma^0\lambda_0^{\mathbf{A}\,(0)} + C^{\mathbf{A}}{}_{\mathbf{C}}\lambda_1^{\mathbf{C}\,(0)}\big) + \\
&\qquad + \lambda_0^{\mathbf{A}\,(0)}\lambda_1^{\mathbf{B}\,(2)} + \lambda_0^{\mathbf{B}\,(0)}\lambda_1^{\mathbf{A}\,(2)} - \sigma^0\bar\sigma^0\big(\lambda_0^{\mathbf{A}\,(0)}\,{}_0\eth'\lambda_0^{\mathbf{B}\,(0)} + \lambda_0^{\mathbf{B}\,(0)}\,{}_0\eth'\lambda_0^{\mathbf{A}\,(0)}\big)\Big)\mathrm{d}\mathcal{S} = \\
&= \oint_{\mathcal{S}}\Big(\lambda_0^{\mathbf{A}\,(0)}\big(-{}_0\eth'\lambda_0^{\mathbf{B}\,(2)} + \lambda_1^{\mathbf{B}\,(2)}\big) + \lambda_0^{\mathbf{B}\,(0)}\big(-{}_0\eth'\lambda_0^{\mathbf{A}\,(0)} + \lambda_1^{\mathbf{A}\,(2)}\big) - \sigma^0\bar\sigma^0\,{}_0\eth\big(\lambda_0^{\mathbf{A}\,(0)}\lambda_0^{\mathbf{B}\,(0)}\big) - \\
&\qquad - \big(C^{\mathbf{A}}{}_{\mathbf{C}}\delta^{\mathbf{B}}_{\mathbf{D}} + C^{\mathbf{B}}{}_{\mathbf{D}}\delta^{\mathbf{A}}_{\mathbf{C}}\big)\,{}_0\eth\bar\sigma^0\lambda_0^{\mathbf{C}\,(0)}\lambda_0^{\mathbf{D}\,(0)} + C^{\mathbf{A}}{}_{\mathbf{C}}C^{\mathbf{B}}{}_{\mathbf{D}}\big(\lambda_0^{\mathbf{C}\,(0)}\lambda_1^{\mathbf{D}\,(0)} + \lambda_1^{\mathbf{C}\,(0)}\lambda_0^{\mathbf{D}\,(0)}\big)\Big)\mathrm{d}\mathcal{S} =
\end{aligned}
$$

$$= \oint_{\mathcal{S}} \Big(\lambda_0^{\mathbf{A}(0)} \big(- {}_0\eth' \lambda_0^{\mathbf{B}(2)} + \lambda_1^{\mathbf{B}(2)} \big) + \lambda_0^{\mathbf{B}(0)} \big(- {}_0\eth' \lambda_0^{\mathbf{A}(2)} + \lambda_1^{\mathbf{A}(2)} \big) + {}_0\eth' \big(\sigma^0 \bar\sigma^0 \big) \lambda_0^{\mathbf{A}(0)} \lambda_0^{\mathbf{B}(0)} \Big) \mathrm{d}\mathcal{S} =$$
$$= \oint_{\mathcal{S}} \Big(\bar\psi_{1'}{}^0 + 2\bar\sigma^0 \, {}_0\eth' \sigma^0 + {}_0\eth' \big(\sigma^0 \bar\sigma^0 \big) \Big) \lambda_0^{\mathbf{A}(0)} \lambda_0^{\mathbf{B}(0)} \mathrm{d}\mathcal{S}, \qquad (6.4)$$

which is precisely Bramson's specific spin-angular momentum expression based on the asymptotic twistor equation. Thus, substituting $\sigma^0 = -{}_0\eth^2 S$ here, finally we obtain (5.1), i.e. *for stationary spacetimes the holomorphic and the anti-holomorphic constructions give the same 'standard' expression.* We expect that at the *past* null infinity the holomorphic construction works properly, and the anti-holomorphic diverges in presence of incoming gravitational radiation.

Appendix: Special spin frames and asymptotic symmetries of $\mathcal{I}^+$

On the bundle $\mathbf{S}^A(\mathcal{S})$ of unprimed spinors over $\mathcal{S}$ two connections can be introduced in a natural way by means of the spacetime connection. The first is the simple projection to $\mathcal{S}$ of the spacetime covariant derivative operator, $\Delta_e := \Pi^f_e \nabla_f$, where the orthogonal projection to $\mathcal{S}$ can be given e.g. by $\Pi^a_b = -m^a \bar m_b - \bar m^a m_b$. The other is the spinor form of the covariant derivative operator δ_e defined on any spacetime vector field X^a by $\delta_e X^a := \Pi^a_b \Delta_e (\Pi^b_c X^c) + (\delta^a_b - \Pi^a_b) \Delta_e ((\delta^b_c - \Pi^b_c) X^c)$. The difference of these two connections is essentially the extrinsic curvature of $\mathcal{S}$ in the spacetime. (For the details see e.g. [50,51].)

Representing the spinor bundle by the line bundles $E(p,q)$, $p - q \in \mathbf{Z}$, the derivative operator δ_e can be represented by the edth and edth-prime operators $\eth$ and $\eth'$ [52]. Explicitly, by the isomorphism $\mathbf{S}_A(\mathcal{S}) \approx E(1,0) \oplus E(-1,0)$, $\lambda_A \approx (\lambda_0, \lambda_1)$, the derivatives $m^e \delta_e \lambda_A$ and $\bar m^e \delta_e \lambda_A$ are represented by the cross sections $(\eth\lambda_0, \eth\lambda_1)$ and $(\eth'\lambda_0, \eth'\lambda_1)$, respectively. Denoting e.g. $\eth$ acting on the cross sections of $E(p,q)$ by $\eth_{(p,q)}$, on topological 2-spheres $\dim\ker \eth_{(p,p+n)} = \dim\ker \eth'_{(p+n,p)} = 0$ for any $p \in \mathbf{R}$ and $n \in \mathbf{N}$, and $\dim\ker \eth_{(p+n,p)} = \dim\ker \eth'_{(p,p+n)} = 1 + 2n$ for any $p \in \mathbf{R}$ and $n = 0, 1, 2, ...$ (For the corresponding kernels on higher genus surfaces see [50].) The irreducible parts of Δ_e and δ_e, respectively, are

$$\begin{aligned}
\mathcal{T}^-(\lambda_0,\lambda_1) &:= m^e \big(\Delta_e \lambda_A\big) o^A = \eth\lambda_0 + \sigma\lambda_1, & t^-(\lambda_0,\lambda_1) &:= m^e \big(\delta_e\lambda_A\big) o^A = \eth\lambda_0, & (A.0.1.a,b)\\
\Delta^+(\lambda_0,\lambda_1) &:= \bar m^e \big(\Delta_e \lambda_A\big) o^A = \eth'\lambda_0 + \rho\lambda_1, & \delta^+(\lambda_0,\lambda_1) &:= \bar m^e \big(\delta_e\lambda_A\big) o^A = \eth'\lambda_0, & (A.0.2.a,b)\\
-\Delta^-(\lambda_0,\lambda_1) &:= m^e \big(\Delta_e \lambda_A\big) \iota^A = \eth\lambda_1 + \rho'\lambda_0, & -\delta^-(\lambda_0,\lambda_1) &:= m^e \big(\delta_e\lambda_A\big) \iota^A = \eth\lambda_1, & (A.0.3.a,b)\\
-\mathcal{T}^+(\lambda_0,\lambda_1) &:= \bar m^e \big(\Delta_e \lambda_A\big) \iota^A = \eth'\lambda_1 + \sigma'\lambda_0, & -t^+(\lambda_0,\lambda_1) &:= \bar m^e \big(\delta_e\lambda_A\big) \iota^A = \eth'\lambda_1. & (A.0.4.a,b)
\end{aligned}$$

In particular, a spinor field λ_A is constant on $\mathcal{S}$ with respect to Δ_e iff $(\lambda_0,\lambda_1) \in \ker(\mathcal{T}^- \oplus \Delta^+ \oplus \Delta^- \oplus \mathcal{T}^+)$. As Bramson showed [53], such a spinor field does not exist even on large spheres near the future or past null infinity in a general asymptotically flat spacetime; and for a finite topological 2-sphere, being the boundary of some compact spacelike hypersurface Σ on which the dominant energy condition is satisfied, the existence of a constant spinor field implies that $D(\Sigma)$ must have a *pp*-wave metric [46,47]. To weaken the notion of the Δ_e-constant spinor fields on $\mathcal{S}$ or on the cuts of $\mathcal{I}^+$, in principle there are six natural possibilities. These are represented by the kernel of the first order operators (see [47]) $\Delta := \Delta^+ \oplus \Delta^-$, $\mathcal{H}^- := \Delta^- \oplus \mathcal{T}^-$, $\mathcal{H}^+ := \Delta^+ \oplus \mathcal{T}^+$, $\mathcal{C}^+ := \Delta^+ \oplus \mathcal{T}^-$, $\mathcal{C}^- := \Delta^- \oplus \mathcal{T}^+$ and $\mathcal{T} := \mathcal{T}^- \oplus \mathcal{T}^+$. Similarly, λ_A is constant with respect to δ_e iff $(\lambda_0,\lambda_1) \in \ker(t^- \oplus \delta^+ \oplus \delta^- \oplus t^+)$. However, it is easy to see that the existence of a nontrivial δ_e-constant spinor field implies the vanishing of the curvature of δ_e, and then, via e.g. the Gauss–Bonnet theorem, that the 2-surface must be a torus (see [50]). Thus if we want to weaken the condition $\delta_e \lambda_A = 0$, we naturally arrive at the first order operators $\delta := \delta^+ \oplus \delta^-$, $h^- := \delta^- \oplus t^-$, $h^+ := \delta^+ \oplus t^+$, $c^+ := \delta^+ \oplus t^-$,

$c^- := \delta^- \oplus t^+$ and $t := t^- \oplus t^+$. (This δ should not be confused with the operator $\delta := m^a\nabla_a$ of the Newman–Penrose formalism introduced in section 3.)

For round spheres of area-radius r (i.e. for a metric sphere in a spherically symmetric spacetime whose radius r is defined by $4\pi r^2 := \mathrm{Area}(\mathcal{S})$) one has $\sigma = 0 = \sigma'$, the convergences ρ and ρ' are constant on $\mathcal{S}$, and $\eth = \frac{1}{r}\,{}_0\eth$ and $\eth' = \frac{1}{r}\,{}_0\eth'$. In particular, in Minkowski spacetime $\rho = -\frac{1}{r}$ and $\rho' = \frac{1}{2r}$. By (3.2-7) and (3.9), asymptotically, near the future null infinity, the operators (A.0.1.a-4.a) tend to ${}_\infty\mathcal{T}^-(\lambda_0,\lambda_1) := {}_0\eth\lambda_0$, ${}_\infty\Delta^+(\lambda_0,\lambda_1) := {}_0\eth'\lambda_0 - \lambda_1$, $-{}_\infty\Delta^-(\lambda_0,\lambda_1) := {}_0\eth\lambda_1 + \frac{1}{2}\lambda_0$ and $-{}_\infty\mathcal{T}^+(\lambda_0,\lambda_1) := {}_0\eth'\lambda_1 - \dot{\bar\sigma}^0\lambda_0$, respectively. The analogous ${}_\infty\delta^\pm$ and ${}_\infty t^\pm$ are just the corresponding unit sphere edth and edth-prime operators. In the rest of this appendix we discuss the direct sum operators briefly, calculate their kernels explicitly for round spheres, and determine how they are related to the asymptotic symmetries of the spacetime at the future null infinity. Although at the quasi-local level only $\mathcal{H}^\pm$ yield acceptable spinor fields for (2.2) and (2.3) [47], we will see that asymptotically, near the future null infinity any of h^-, $\mathcal{H}^-$, $\mathcal{C}^+$, $\mathcal{T}$ and t (and 'at' infinity ${}_\infty\Delta$ also) can be used to recover the asymptotic symmetries.

A.1 The Dirac–Witten operators Δ and δ

By the definitions $\Delta_{A'A}\lambda^A = \bar\iota_{A'}\Delta^+(\lambda_0,\lambda_1) - \bar o_{A'}\Delta^-(\lambda_0,\lambda_1)$ and $\delta_{A'A}\lambda^A = \bar\iota_{A'}\delta^+(\lambda_0,\lambda_1) - \bar o_{A'}\delta^-(\lambda_0,\lambda_1)$; i.e. $\Delta(\lambda_0,\lambda_1)=0$ and $\delta(\lambda_0,\lambda_1)=0$ are just the Dirac–Witten equations on $\mathcal{S}$ with respect to Δ_e and δ_e, respectively. They are elliptic operators with vanishing index, and hence in the generic case their kernel is zero dimensional. In fact, by $\dim\ker\eth_{(-1,0)} = \dim\ker\eth'_{(1,0)} = 0$ one has $\dim\ker\delta = 0$. On the other hand, there might be exceptional 2-surfaces, even among the round spheres, for which $\dim\ker\Delta$ is not zero. To find these exceptional round 2-spheres first take the $\eth$-derivative of $\Delta^+(\lambda_0,\lambda_1)=0$ and the $\eth'$-derivative of $\Delta^-(\lambda_0,\lambda_1)=0$. We obtain $\eth\eth'\lambda_0 = \rho\rho'\lambda_0$ and $\eth'\eth\lambda_1 = \rho\rho'\lambda_1$, i.e. λ_0 and λ_1 are eigenfunctions of $\eth\eth'$ and $\eth'\eth$, respectively, with the same eigenvalue $\rho\rho'$. Thus let us expand them as series of the $s=\pm\frac{1}{2}$ spin weighted spherical harmonics: $\lambda_0 = \sum_{j,m} c_0^{j,m}\,{}_{\frac{1}{2}}Y_{j,m}$ and $\lambda_1 = \sum_{j,m} c_1^{j,m}\,{}_{-\frac{1}{2}}Y_{j,m}$, where $j = \frac{1}{2}, \frac{3}{2}, ...$ and $m = -j, -j+1, ..., j$ (see e.g. [40]). Recalling that $\eth\,{}_sY_{j,m} = -\frac{1}{\sqrt{2}r}\sqrt{(j+s+1)(j-s)}\,{}_{s+1}Y_{j,m}$ and $\eth'\,{}_sY_{j,m} = \frac{1}{\sqrt{2}r}\sqrt{(j-s+1)(j+s)}\,{}_{s-1}Y_{j,m}$, the second order equations yield that

$$2r^2\rho\rho' = -(j+\frac{1}{2})^2, \qquad j = \frac{1}{2}, \frac{3}{2}, ... \tag{A.1.1}$$

i.e. *the Dirac–Witten operator Δ can have non-trivial kernel only for discrete values of* $2r^2\rho\rho'$. Thus for a given, allowed $\rho\rho'$ labelled by j one has $\lambda_0 = \sum_m c_0^m\,{}_{\frac{1}{2}}Y_{j,m}$ and $\lambda_1 = \sum_m c_1^m\,{}_{-\frac{1}{2}}Y_{j,m}$. Substituting these into the first order equations we finally get $c_1^m = \sqrt{2}(j+\frac{1}{2})^{-1}r\rho'\,c_0^m$, i.e.

$$\lambda^A = \sum_{m=-j}^{j} c_0^m\Big(\frac{\sqrt{2}}{j+\frac{1}{2}}r\rho'\,{}_{-\frac{1}{2}}Y_{j,m}\,o^A - {}_{\frac{1}{2}}Y_{j,m}\,\iota^A\Big). \tag{A.1.2}$$

In particular, for $j=\frac{1}{2}$ (A.1.1) is just the condition that the round sphere is a metric sphere in Minkowski spacetime, whenever (A.1.2) is the combination of the restriction to $\mathcal{S}$ of the two constant spinor fields given explicitly by (3.1). (For the explicit expression of ${}_sY_{j,m}$, see e.g. [40,54].) However, apart from the Minkowski case, (A.1.1) yields *negative* Hawking energy $E_H := (\mathrm{Area}(\mathcal{S}_r)/16\pi G^2)^{\frac{1}{2}}(1+\frac{1}{2\pi}\oint_{\mathcal{S}_r}\rho\rho'\mathrm{d}\mathcal{S}_r) = \frac{r}{2G}(1-(j+\frac{1}{2})^2)$, which is just the (holomorphic and anti-holomorphic) Dougan–Mason energy in an appropriate basis, because the mass is $m^2 := \varepsilon_{\mathbf{AB}}\varepsilon_{\mathbf{A'B'}}P^{\mathbf{AA'}}P^{\mathbf{BB'}} = \frac{r^2}{4G^2}(1+2r^2\rho\rho')^2$. Thus, in general, the operators Δ and δ do not define special spinor fields on round spheres.

On the other hand, on *large spheres* near the future null infinity the Dirac–Witten equations tend to ${}_\infty\Delta(\lambda_0,\lambda_1)=0$, the Dirac–Witten equations on the unit round sphere in Minkowski spacetime. But ${}_\infty\Delta(\lambda_0,\lambda_1)=0$ admits $\mathcal{E}_0^A = O^A$ and $\mathcal{E}_1^A = I^A$, given explicitly by (3.1), as independent solutions. Therefore,

by the analysis of Section 3, both the BMS translation and (up to supertranslations) the BMS boost-rotation generators can be recovered from the (exceptional) solutions of the Dirac–Witten equations on the cuts of $\mathscr{I}^+$. However, apart from the Minkowski case, these solutions are *not* limits of solutions of the Dirac–Witten equations on finite, large spheres, because the latter's do not exist in general.

A.2 The anti-holomorphy operators $\mathcal{H}^-$ and h^-

By the definitions $m^e\Delta_e\lambda_A = -o_A\Delta^-(\lambda_0,\lambda_1) - \iota_A\mathcal{T}^-(\lambda_0,\lambda_1)$ and $m^e\delta_e\lambda_A = -o_A\delta^-(\lambda_0,\lambda_1) - \iota_A t^-(\lambda_0,\lambda_1)$; i.e. λ_A is anti-holomorphic with respect to Δ_e or δ_e iff $\mathcal{H}^-(\lambda_0,\lambda_1) = 0$ or $h^-(\lambda_0,\lambda_1) = 0$, respectively. The symplectic scalar product of any two anti-holomorphic spinor fields (with respect to either Δ_e or δ_e) is anti-holomorphic, and hence constant on $\mathcal{S}$. $\mathcal{H}^-$ and h^- are elliptic, and their index is $2(1-g)$, where g is the genus of $\mathcal{S}$. Therefore, in the generic case on topological 2-spheres their kernel is two complex dimensional.

In fact, by $\dim\ker\eth_{(-1,0)} = 0$ and $\dim\ker\eth_{(1,0)} = 2$ the δ_e-anti-holomorphic spinor fields have the form $\lambda_A = -\lambda_0\iota_A$, where $\lambda_0 \in \ker\eth_{(1,0)}$. Therefore, the space of δ_e-anti-holomorphic spinor fields does not inherit a natural $SL(2,\mathbf{C})$ scalar product from ε_{AB}. However, *on round spheres* these spinor fields can be normalized with respect to each other: For α_A, $\tilde\alpha_A \in \ker h^-$ the combination $\alpha_0\bar\alpha_{0'} + \tilde\alpha_0\bar{\tilde\alpha}_{0'}$ is constant on $\mathcal{S}$, and can be required to be $\sqrt{2}$. In fact, on round spheres for the two independent explicit solutions we can choose

$$\alpha^0_A = \frac{\mathrm{i}\sqrt[4]{2}\,\zeta}{\sqrt{1+\zeta\bar\zeta}}\,\iota_A, \qquad \alpha^1_A = \frac{\mathrm{i}\sqrt[4]{2}}{\sqrt{1+\zeta\bar\zeta}}\,\iota_A, \tag{A.2.1}$$

which are normalized in this sense. The transformations leaving this normalization invariant is $SL(2,\mathbf{C})\times U(1)$. (In fact, the normalization condition is only *one real* condition, and this leaves an unspecified phase in the basis solutions (A.2.1).) The spinor components $\alpha^{\mathbf{A}}_A o^A$ are proportional to the $s=\frac{1}{2}$ spin weighted spherical harmonics: They are $-\sqrt[4]{2}\sqrt{2\pi}\,{}_{\frac{1}{2}}Y_{\frac{1}{2},\pm\frac{1}{2}}$, respectively, and their normalization with respect to the unit sphere volume form is given by $\oint_{\mathcal{S}}(\alpha^{\mathbf{A}}_A o^A)(\bar\alpha^{\mathbf{B}'}_{B'}\bar o^{B'})\mathrm{d}\mathcal{S} = 2\sqrt{2}\pi\,\mathrm{diag}(1,1) = 4\pi\,\sigma_0^{\mathbf{A}\mathbf{B}'}$. Since *on large spheres* the equations for the δ_e-anti-holomorphic spinor fields are just those on the round unit sphere, the contravariant form of the independent δ_e-anti-holomorphic spinor fields on cuts of $\mathscr{I}^+$ are given by $\alpha^A_{\mathbf{A}} := \varepsilon^{AB}\alpha^{\mathbf{B}}_B\varepsilon_{\mathbf{B}\mathbf{A}}$; i.e. explicitly by $\alpha^A_0 = -\mathrm{i}\sqrt[4]{2}(1+\zeta\bar\zeta)^{-\frac{1}{2}}\iota^A$ and $\alpha^A_1 = \mathrm{i}\sqrt[4]{2}\zeta(1+\zeta\bar\zeta)^{-\frac{1}{2}}\iota^A$. Comparing these with the functions $\tau_{\mathbf{A}}$ at the end of the first paragraph in Section 3, one can see that $\alpha^A_{\mathbf{A}} = \tau_{\mathbf{A}}\iota^A = \tau_{\mathbf{A}}\hat\iota^A$, *which are just the leading parts of the spinor constituents of the BMS translations at* $\mathscr{I}^+$. The main part of the anti-self-dual BMS rotations can also be expressed as $-o_Ao_B\alpha^A_{(\mathbf{A}}\alpha^B_{\mathbf{B})}$. These δ_e-anti-holomorphic spinor fields are used in [55] to find a relationship between the Bondi–Sachs mass at the *past* null infinity and the area of a marginally trapped surface.

For the independent Δ_e-anti-holomorphic spinor fields *on round spheres* we choose

$$\nu^0_A = -\frac{\mathrm{i}}{\sqrt[4]{2}}\frac{1}{\sqrt{1+\zeta\bar\zeta}}\Big(2r\rho' o_A - \sqrt{2}\zeta\iota_A\Big), \qquad \nu^1_A = \frac{\mathrm{i}}{\sqrt[4]{2}}\frac{1}{\sqrt{1+\zeta\bar\zeta}}\Big(\bar\zeta o_A + \frac{1}{\sqrt{2}r\rho'}\iota_A\Big). \tag{A.2.2}$$

These are normalized with respect to the pointwise $SL(2,\mathbf{C})$ scalar product: $\varepsilon^{AB}\nu^{\mathbf{A}}_A\nu^{\mathbf{B}}_B = \varepsilon^{\mathbf{A}\mathbf{B}}$. Therefore, a natural spin space structure is inherited on the space of Δ_e-anti-holomorphic spinor fields on round spheres. The L_2 scalar product of the spinor components are $\oint_{\mathcal{S}}\nu^{\mathbf{A}}_0\bar\nu^{\mathbf{B}'}_{0'}\mathrm{d}\mathcal{S} = 2\sqrt{2}\pi\,\mathrm{diag}(1,(2r\rho')^{-2})$ and $\oint_{\mathcal{S}}\nu^{\mathbf{A}}_1\bar\nu^{\mathbf{B}'}_{1'}\mathrm{d}\mathcal{S} = \sqrt{2}\pi\,\mathrm{diag}((2r\rho')^2,1)$. In Minkowski spacetime $\nu^{\mathbf{A}}_A$ reduces to the restriction to $\mathcal{S}$ of $-\mathcal{E}^{\mathbf{A}}_A = \{I_A, -O_A\}$, minus the dual of the Cartesian spin frame (3.1). Since the equations for the Δ_e-anti-holomorphic spinor fields *on large spheres* are the ones on the round unit sphere in Minkowski spacetime, *the spinor constituents of the BMS translations on the cuts of* $\mathscr{I}^+$ *are given by the components of the* Δ_e*-anti-holomorphic spinor fields via* $\tau_{\mathbf{A}} = o_A\mathcal{E}^A_{\mathbf{A}} = o_A\varepsilon^{AB}\nu^{\mathbf{B}}_B\varepsilon_{\mathbf{B}\mathbf{A}}$*, and hence the spinor constituents themselves are* $\varepsilon^{AB}\nu^{\mathbf{B}}_B\varepsilon_{\mathbf{B}\mathbf{A}} = \tau_{\mathbf{A}}\hat\iota^A + O(\Omega)$.

This representation of the BMS translations is used in [35]. The expression of the main part of the anti-self-dual BMS rotations by the Δ_e-anti-holomorphic spinors is then obvious.

A.3 The holomorphy operators $\mathcal{H}^+$ and h^+

As a simple consequence of the definitions, λ_A is holomorphic with respect to Δ_e or to δ_e iff $\mathcal{H}^+(\lambda_0,\lambda_1)=0$ or $h^+(\lambda_0,\lambda_1)=0$, respectively. The properties of these differential operators are similar to those of $\mathcal{H}^-$ and h^-, thus we concentrate only the differences. By $\dim\ker\eth'_{(1,0)}=0$ and $\dim\ker\eth'_{(-1,0)}=2$ the δ_e-holomorphic spinor fields have the form $\lambda_A=\lambda_1 o_A$, where $\lambda_1\in\ker\eth'_{(-1,0)}$. *On round spheres* for the two independent explicit solutions we can choose

$$\beta^0_A=-\frac{\mathrm{i}}{\sqrt[4]{2}}\frac{1}{\sqrt{1+\zeta\bar\zeta}}\,o_A,\qquad \beta^1_A=\frac{\mathrm{i}}{\sqrt[4]{2}}\frac{\bar\zeta}{\sqrt{1+\zeta\bar\zeta}}\,o_A, \tag{A.3.1}$$

whose components $\beta^{\mathbf{A}}_A\iota^A$ are proportional to the $s=-\frac{1}{2}$ spin weighted spherical harmonics: They are $-\sqrt[4]{2}\sqrt{\pi}\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},\pm\frac{1}{2}}$, respectively. Their contravariant form are the spinor constituents of the BMS translations on the *past* null infinity $\mathcal{I}^-$, and *not* on $\mathcal{I}^+$. *On round spheres*

$$\mu^0_A=\frac{\mathrm{i}}{\sqrt[4]{2}}\frac{1}{\sqrt{1+\zeta\bar\zeta}}\Bigl(\frac{1}{r\rho}o_A+\zeta\sqrt{2}\iota_A\Bigr),\qquad \mu^1_A=\frac{\mathrm{i}}{\sqrt[4]{2}}\frac{1}{\sqrt{1+\zeta\bar\zeta}}\Bigl(\bar\zeta o_A-r\rho\sqrt{2}\iota_A\Bigr) \tag{A.3.2}$$

form a normalized spin frame in the space of Δ_e-holomorphic spinor fields, which in Minkowski spacetime reduce to $\{I_A,-O_A\}$. However, the Δ_e-holomorphic spinor equations *on large spheres* near the *future* null infinity do *not* reduce to the ones on a round sphere: Because of the $\dot{\bar\sigma}^0$ term in ${}_\infty\mathcal{T}^+$ the solutions of ${}_\infty\mathcal{H}^+(\lambda_0,\lambda_1)=0$ are *not* the restriction to $\mathcal{S}$ of the constant spinor fields of the Minkowski spacetime. They are constant only if $\dot\sigma^0=0$, i.e. in absence of outgoing gravitational radiation.

A.4 The operators $\mathcal{C}^+$ and c^+

By the definitions the equations $\mathcal{C}^\pm(\lambda_0,\lambda_1)=0$ and $c^\pm(\lambda_0,\lambda_1)=0$ can be written into the manifestly covariant form $(\Delta_a\lambda_B)\pi^{\pm B}{}_C=0$ and $(\delta_a\lambda_B)\pi^{\pm B}{}_C=0$, respectively, where $\pi^{+A}{}_B=\iota^Ao_B$ and $\pi^{-A}{}_B=-o^A\iota_B$ are the projections of the spin spaces onto the space of the ±1 eigenspinors of the spinor $\gamma^A{}_B=o^A\iota_B+\iota^Ao_B$, defining a GHP-spin frame independent notion of chirality on the spin spaces (see [51]). $\mathcal{C}^\pm$ and $c^\pm$ are not elliptic operators. In fact, by $\dim\ker\eth_{(1,0)}=2$ and $\dim\ker\eth'_{(1,0)}=0$ on round spheres the kernel of $\mathcal{C}^+$ is two dimensional only if $\rho\neq0$, otherwise $\dim\ker\mathcal{C}^+=\infty$. In particular, c^+ is just the $\rho=0$ special case of $\mathcal{C}^+$, whenever the elements of c^+ have the form $\lambda_A=\lambda_1o_A$ with arbitrary $\lambda_1:\mathcal{S}\to\mathbf{R}$. *On round spheres* with non-zero ρ the kernel of $\mathcal{C}^+$ is just the kernel of $\mathcal{H}^+$; i.e. for the two explicit solutions we can choose (A.3.2). However, the equations $\mathcal{C}^+(\lambda_0,\lambda_1)=0$ *on large spheres* near the *future* null infinity tend to the asymptotic twistor equations of Bramson [53], which turn out to be the equations $\mathcal{C}^+(\lambda_0,\lambda_1)=0$ *on the round unit sphere* in Minkowski spacetime, and hence $\ker{}_\infty\mathcal{C}^+=\ker{}_\infty\mathcal{H}^+$ still holds. Therefore, *the ι^A-components of the normalized solutions of the asymptotic twistor equation reproduce the components $\tau_{\mathbf{A}}$ of the spinor constituents of the BMS translations on $\mathcal{I}^+$, and the solutions themselves are* $\varepsilon^{AB}\mu^{\mathbf{B}}_B\varepsilon_{\mathbf{BA}}=\tau_{\mathbf{A}}\hat\iota^A+O(\Omega)$. This is one of the most popular representation of the BMS translations (see [22-24,34,38,40,56]). The expression of the BMS rotations is then straightforward.

A.5 The operators $\mathcal{C}^-$ and c^-

By $\dim\ker\eth_{(-1,0)}=0$ and $\dim\ker\eth'_{(-1,0)}=2$ the kernel of $\mathcal{C}^-$ *on round spheres* is two dimensional only if $\rho'\neq0$, otherwise $\dim\ker\mathcal{C}^-=\infty$. c^- is the $\rho'=0$ special case of $\mathcal{C}^-$, and the elements of c^- have the form

$\lambda_A=-\lambda_0\iota_A$ with arbitrary $\lambda_0:\mathcal{S}\to\mathbf{R}$. If $\rho'\neq 0$ then $\ker\mathcal{C}^-=\ker\mathcal{H}^-$, thus for the two explicit solutions of $\mathcal{C}^-(\lambda_0,\lambda_1)=0$ we can choose (A.2.2). However, the equations $\mathcal{C}^-(\lambda_0,\lambda_1)=0$ *on large spheres* near the *future* null infinity are *not* the equations $\mathcal{C}^-(\lambda_0,\lambda_1)=0$ on the round unit sphere in Minkowski spacetime, just because of the presence of the $\dot{\bar\sigma}^0$ term in ${}_\infty\mathcal{T}^+$. Therefore, the solutions of $\mathcal{C}^-(\lambda_0,\lambda_1)=0$ on large spheres can be used to represent the BMS translations on the *past* null infinity.

A.6 The 2-surface twistor operators $\mathcal{T}$ and t

The 2-surface twistor equations $\mathcal{T}(\lambda_0,\lambda_1)=0$ can be written into the covariant form $\mathcal{T}_{E'EA}{}^B\lambda_B:=\Delta_{E'(E}\lambda_{A)}+\frac{1}{2}\gamma_{EA}\gamma^{BC}\Delta_{E'B}\lambda_C=0$; i.e. the vanishing of the γ_{AB}-trace-free symmetrized Δ_e-derivative of λ_A (see [45]). Similarly, $t(\lambda_0,\lambda_1)=0$ is equivalent to $\delta_{E'(E}\lambda_{A)}+\frac{1}{2}\gamma_{EA}\gamma^{BC}\delta_{E'B}\lambda_C=0$. $\mathcal{T}$ and t are elliptic operators with index $4(1-g)$. *On round spheres* they coincide, and by $\dim\ker\eth_{(1,0)}=\dim\ker\eth'_{(-1,0)}=2$ their kernel is four dimensional. Therefore, the general explicit solution of the 2-surface twistor equations has the form $\omega_A=a_{\mathbf{A}}\alpha^{\mathbf{A}}_A+b_{\mathbf{A}}\beta^{\mathbf{A}}_A$ for complex constants $a_{\mathbf{A}}$, $b_{\mathbf{A}}$, where $\alpha^{\mathbf{A}}_A$ and $\beta^{\mathbf{A}}_A$ are given by (A.2.1) and (A.3.1), respectively. Introducing $\pi_{A'}:=\mathrm{i}\,\Delta_{A'A}\omega^A$ and the 2-surface twistor $\mathtt{Z}^\alpha:=(\omega^A,\pi_{A'})$, the usual Hermitian pointwise scalar product of any two 2-surface twistors $\mathtt{Z}^\alpha$ and $\mathtt{Z}'^\alpha$ is constant on $\mathcal{S}$, and hence defines the familiar Hermitian scalar product on the space $\mathbf{T}^\alpha$ of the 2-surface twistors. However, the flat spacetime definition of the infinity twistor does not define any (global) twistor on $\mathbf{T}^\alpha$, because $\varepsilon^{A'B'}\pi_{A'}\pi'_{B'}$ is not constant on $\mathcal{S}$. Its modification, $\mathtt{I}_{\alpha\beta}\mathtt{Z}^\alpha\mathtt{Z}'^\beta:=\varepsilon^{A'B'}\pi_{A'}\pi'_{B'}+\frac{1}{2r^2}(1+2r^2\rho\rho')\varepsilon_{AB}\omega^A\omega'^B$ is constant on $\mathcal{S}$, but, apart from the Minkowski case $2r^2\rho\rho'=-1$, its rank is four and hence it fails to be simple. Thus, even on general round spheres, $\mathtt{I}_{\alpha\beta}\mathtt{Z}^\beta=0$ cannot be used to *define* 'translations' in the twistor space $\mathbf{T}^\alpha$. The primary parts of the 2-surface twistors defined by t and $\mathcal{T}$ coincide, but their secondary parts do not. The 2-surface twistors defined by t can be recovered from those of $\mathcal{T}$ as the $\rho=0=\rho'$ special case.

On *large spheres* near $\mathcal{I}^+$ the solutions of the 2-surface twistor equations have the structure $\omega_A=a_{\mathbf{A}}\alpha^{\mathbf{A}}_A+b_{\mathbf{A}}\beta^{\mathbf{A}}_A+a_{\mathbf{A}}\gamma^{\mathbf{A}}o_A=(o_C\varepsilon^{CB}\alpha^{\mathbf{A}}_Ba_{\mathbf{A}})\iota_A+(b_{\mathbf{A}}\beta^{\mathbf{A}}_B\iota^B+a_{\mathbf{A}}\gamma^{\mathbf{A}})o_A$, where $\gamma^{\mathbf{A}}$ are of spin weight $-\frac{1}{2}$ and satisfy ${}_0\eth'\gamma^{\mathbf{A}}=\dot{\bar\sigma}^0\alpha^{\mathbf{A}}_0$. Comparing this with the results of subsection **A.2** we find that $\omega^A=a_{\mathbf{A}}\tau^{\mathbf{A}}\iota^A+(b_{\mathbf{A}}\beta^{\mathbf{A}}_1+a_{\mathbf{A}}\gamma^{\mathbf{A}})o^A=a_{\mathbf{A}}\tau^{\mathbf{A}}\hat\iota^A+O(\Omega)$; i.e. the primary part of the 2-surface twistors $\mathtt{Z}^\alpha=(\omega^A,\mathrm{i}\,\Delta_{A'B}\omega^B)$ are just the spinor constituents of the BMS translations. Although the 2-surface twistors on large spheres still form a 4 dimensional complex vector space, half of them (parametrized by the $b_{\mathbf{A}}$'s) die off asymptotically. In absence of outgoing gravitational radiation ${}_\infty\mathcal{T}$ reduces to the homogeneous ${}_\infty t$, whose solutions are just those of $\mathcal{T}$ and of t on the unit round sphere. Interestingly enough, in the conformal approach of the 2-surface twistor equation on the cuts of $\mathcal{I}^+$ (see [12-20,38,40]) the absence of outgoing gravitational radiation yields the homogeneous equations ${}_0\eth\lambda_0=0$, ${}_0\eth'\lambda_1=0$ only after an appropriate supertranslation.

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